# What Are Mathematical Coincidences (and Why Does It Matter)?

Marc Lange
*University of North Carolina at Chapel Hill*
*mlange@email.unc.edu*

Although all mathematical truths are necessary, mathematicians take certain combinations of mathematical truths to be 'coincidental', 'accidental', or 'fortuitous'. The notion of a 'mathematical coincidence' has so far failed to receive sufficient attention from philosophers. I argue that a mathematical coincidence is not merely an unforeseen or surprising mathematical result, and that being a misleading combination of mathematical facts is neither necessary nor sufficient for qualifying as a mathematical coincidence. I argue that although the components of a mathematical coincidence may possess a common explainer, they have no common explanation; that two mathematical facts have a unified explanation makes their truth non-coincidental. I suggest that any motivation we may have for thinking that there are mathematical coincidences should also motivate us to think that there are mathematical explanations, since the notion of a mathematical coincidence can be understood only in terms of the notion of a mathematical explanation. I also argue that the notion of a mathematical coincidence plays an important role in scientific explanation. When two phenomenological laws of nature are similar, despite concerning physically distinct processes, it may be that any correct scientific explanation of their similarity proceeds by revealing their similarity to be no mathematical coincidence.

## 1. The phenomenon to be saved

All mathematical truths are necessary. Accordingly, there might seem to be no place in mathematics for genuine coincidences.[1] However, mathematicians and non-mathematicians alike sometimes encounter a pair (or more) of mathematical facts, F and G, about which they naturally ask, 'Is it a coincidence that both F and G obtain? Or is it no coincidence?' The fact that F and G both obtain, though necessary,

[1] Except in the technical sense (employed in 'coincidence theory') where, for example, the coincidence set of two functions $f(x)$ and $g(x)$ is the set of $x$'s where $f(x) = g(x)$, as when two curves intersect ('coincide').

Mind, Vol. 119 . 474 . April 2010 
doi:10.1093/mind/fzq013 Advance Access publication 23 July 2010

may nevertheless be coincidental, accidental, fortuitous. (All of these terms appear in the mathematical literature.) Here are four examples that may easily prompt us to wonder whether a given combination of mathematical facts is a coincidence.

My first example concerns the sum of the first $n$ natural numbers: $1 + 2 + \ldots + (n - 1) + n$. There are two cases.

When $n$ is even, we can pair the first and last numbers in the sequence, the second and second-to-last, and so forth. The members of each pair sum to $n + 1$. No number is left unpaired, since $n$ is even. The number of pairs is $n/2$ (which is an integer, since $n$ is even). Hence, the sum is $(n + 1)n/2$.

When $n$ is odd, we can pair the numbers as before, except that the middle number in the sequence is left unpaired. Again, the members of each pair sum to $n + 1$. But now there are $(n - 1)/2$ pairs, since the middle number $(n + 1)/2$ is unpaired. The total sum is then the sum of the paired numbers plus the middle number: $(n + 1)(n - 1)/2 + (n + 1)/2$. This expression simplifies to $(n + 1)n/2$ — remarkably, the same as the formula we just derived for even $n$.

Faced with this proof, we might well wonder: Is it a coincidence that the same formula emerges in both cases? Even if you know the answer to this question, I think you can appreciate how it might seem like a mathematical accident — an algebraic miracle — that the second argument arrives at the same formula as the first. The proof I have just given shows it to be necessary that the same formula applies to even $n$ and odd $n$ — but does not show it to be no coincidence. Indeed, the proof (with its two cases) makes it appear coincidental.

Here is another potential mathematical coincidence — from an article by the mathematician Philip Davis, one of the only two papers I know concerning the concept of a mathematical coincidence.[2] Davis (1981, p. 312) points out that the thirteenth digit of the decimal representation of $\pi$ (= 3.14 159 265 358 979 3 ... ) is the same as the thirteenth digit of the decimal representation of $e$ (= 2.71 828 182 845 904 5 ... ); both are 9. Again: is this a coincidence?

The other paper about mathematical coincidence is a lovely unpublished piece by Roy Sorensen. He takes this nice example from a brief ‘Gleaning’ entitled ‘A Calculator Coincidence?’ appearing on page 283

[2] After writing this paper, I learned that mathematical coincidences are briefly examined in Baker forthcoming (which proposes a view similar in some respects to mine), in Corfield 2005, and in Potter 1993.

of the December 1986 issue of *The Mathematical Gazette*. Take an ordinary calculator keyboard:

| | | |
|---|---|---|
| 7 | 8 | 9 |
| 4 | 5 | 6 |
| 1 | 2 | 3 |

We can form a six-digit number by taking the three digits on any row, column, or main diagonal on the keyboard in forward and then in reverse order. For instance, the bottom row taken from left to right, and then right to left, yields 123321. There are sixteen such numbers: 123321, 321123, 456654, 654456, 789987, 987789, 147741, 741147, 258852, 852258, 369963, 963369, 159951, 951159, 357753, and 753357. As you can easily verify with a calculator, every one of these numbers is divisible by 37. Is this (as the title of the article asks) a coincidence?

Here is a fourth example. Consider these two Diophantine equations (that is, equations where the variables can take only integer values):

$$2x^2(x^2 - 1) = 3(y^2 - 1)$$

and

$$x(x - 1)/2 = 2^n - 1$$

As it happens, each equation has exactly the same five positive solutions: $x = 1, 2, 3, 6$, and 91 (Guy 1988, p. 704). Coincidence?

There is no reason to keep you in suspense. As far as I know, the last example and the one involving the thirteenth digits of $\pi$ and $e$ are genuine mathematical coincidences (despite being necessary). But it turns out to be no coincidence that the same $n(n + 1)/2$ formula yields the sum of the first $n$ natural numbers whether $n$ is even or odd. It is also no coincidence that every number produced by using the calculator keyboard in the manner I described is divisible by 37. In this paper, my main task will be to identify the fundamental difference between mathematical coincidences and non-coincidences.

Their fundamental difference should be reflected in the way in which we would show that a given pair of mathematical facts is no coincidence. Here is an argument showing it to be no coincidence that the same $n(n+1)/2$ formula yields the sum $S$ of the first $n$ natural numbers whether $n$ is even or odd:

> $S = 1 + 2 + \ldots + (n-1) + n$
>
> $S = n + (n-1) + \ldots + 2 + 1$
>
> If we pair the first terms, the second terms, and so forth in each sum, then each pair adds to $(n+1)$, and there are $n$ pairs. So $2S = n(n+1)$, and hence $S = n(n+1)/2$.

This proof is only slightly different from the proof I gave earlier that dealt separately with even $n$ and odd $n$. Yet the earlier proof failed to reveal that the formula's success for both even $n$ and odd $n$ is no coincidence. What allows the second proof to do better than the first one did?[3]

We can pose the same question regarding the calculator keyboard example. A proof that looks at each one of the sixteen numbers individually, showing it to be divisible by 37, merely prompts the question, 'Is it a coincidence that all of the numbers arrived at in this way are divisible by 37?' As Sorensen notes, this question is answered in the negative by another proof of the same result — given by Eric Nummela in a piece entitled 'No Coincidence' that appeared in *The Mathematical Gazette* for June 1987:

> This is *no* coincidence. For let $a$, $a+d$, $a+2d$ be any three integers in arithmetic progression. Then
>
> $$a.10^5 + (a+d).10^4 + (a+2d).10^3 + (a+2d).10^2 + (a+d).10 + a.1$$
> $$= a(10^5 + 10^4 + 10^3 + 10^2 + 10 + 1) + d(10^4 + 2.10^3 + 2.10^2 + 10)$$
> $$= 111111a + 12210d = 1221(91a + 10d).$$
>
> So not only is the number divisible by 37, but by 1221 $(= 3 \times 11 \times 37)$ (Nummela 1987, p. 147)

Again, why does this proof, unlike the earlier one, show it to be no coincidence that all of the calculator-keyboard numbers are divisible by 37?

I think that these two arguments, in showing certain combinations of mathematical facts to be no coincidence, do a great deal to suggest

[3] Steiner (1978a, p. 136) deems the second proof 'more illuminating' than a proof of the summation formula that proceeds by mathematical induction. (Steiner does not discuss mathematical coincidences or contrast the second proof with separate proofs of the even and odd cases (as in my first proof).)

what a mathematical coincidence is. To work up to that suggestion, I shall first consider some deflationary approaches to mathematical coincidence. Having argued against those proposals (in section 2), I will (in section 3) offer my own account. Roughly speaking, I shall argue that if F and G are mathematical truths, then it is a coincidence that both F and G are true if and only if F and G have no common mathematical explanation. Such an explanation is supplied by the second proof I gave of the $n(n+1)/2$ formula for the sum of the first $n$ natural numbers, as well as by Nummela's proof of the regularity involving the calculator-keyboard numbers. No such explanation is available for a genuine mathematical coincidence. Finally (in section 4), I will point out that certain scientific explanations in physics use the same mathematical proof to derive physically unrelated phenomenological laws from distinct but mathematically similar fundamental laws. These scientific explanations explain why those phenomenological laws are so similar; the analogy between those laws turns out to be no mathematical coincidence. The notion of a mathematical coincidence thus has an important role to play even outside of mathematics.

My account of mathematical coincidence presupposes a distinction between a mathematical proof that explains why some theorem holds and a proof that merely proves that it holds. I shall not offer a general account of this distinction here. My target in this paper is mathematical coincidence, not mathematical explanation.[4] However, in arguing that mathematical coincidence is a genuine, important feature of mathematical practice and that it should be understood in terms of mathematical explanation, I am arguing (by a kind of 'inference to the best philosophical explanation') that there is an important distinction between mathematical proofs that explain and proofs that do not.[5]

[4] Some notable papers on the distinction between proofs that explain why a mathematical theorem holds and proofs that merely prove that it holds are Steiner 1978a, Kitcher 1984 (esp. pp. 208–9, 227) and 1989 (esp. pp. 423–6, 437), Resnik and Kushner 1987, Hafner and Mancosu 2005, Tappenden 2005, and Mancosu 2008b. References to other literature on mathematical explanation can be found therein.

[5] Although every mathematical explanation I shall discuss involves a proof, I am not inclined to presuppose that every mathematical explanation is a proof. Perhaps a mathematical fact can sometimes be explained just by being recast in terms of a different conceptual framework, for example.

More than one proof might explain a given mathematical fact, and some mathematical facts may have no explanation at all. (At least, I say nothing here to preclude these possibilities.)

I am also thereby arguing for a criterion of adequacy for any general account of mathematical explanation: that it must be capable of being plugged into my account of mathematical coincidence.

## 2. Is 'mathematical coincidence' an epistemic notion?

Sorensen proposes that the distinction between coincidences and non-coincidences in mathematics is epistemic, so that from a mathematically omniscient viewpoint, there are no mathematical coincidences. I shall now examine several ways in which this idea might be elaborated, arguing that none of them captures the distinction.

We might try to define a 'mathematical coincidence' as an unforeseen or surprising mathematical result. Certainly, all four of the examples given in the previous section are surprising and difficult to foresee in advance of being proved. Moreover, even given part of each result (for example, that the sum of the first $n$ numbers, when $n$ is even, is $n(n+1)/2$; that one of the calculator-keyboard numbers is divisible by 37; that $2x^2(x^2-1) = 3(y^2-1)$ is solved by exactly $x = 1$, 2, 3, 6, and 91; that 9 is the thirteenth digit of $e$), the rest of that result is still not expected. However, as we saw, two of these results are mathematical coincidences, while the other two are not. Therefore, the 'unforeseen or surprising fact' proposal does not manage to capture the phenomenon of mathematical coincidence (although, I shall ultimately argue, there remains a grain of truth in it).

A suggestion more like Sorensen's is that a mathematical coincidence is a mathematical fact that misleads us, provoking expectations that turn out to be false. Some of the mathematical coincidences that Sorensen mentions certainly provoke false expectations, such as that $3^2 + 4^2 = 5^2$ and $3^3 + 4^3 + 5^3 = 6^3$. (Contrary to the expectations thus provoked, $3^4 + 4^4 + 5^4 + 6^4 = 2258 \neq 2401 = 7^4$.) Likewise, when a mathematical fact provokes expectations that turn out to be true, we might then be inclined to say that the fact is no coincidence. For instance, is it a coincidence that 25 has at least as many divisors of the form $4k+1$ (1, 5, and 25) as of the form $4k-1$ (none), and that the same goes for 21 (1, and 21; 3 and 7)? No, it is no coincidence — these two examples

Furthermore, the distinction between proofs that explain and proofs that do not may be context sensitive in various ways. (See nn. 17 and 19.) Although I shall ultimately emphasize the way that the components of a non-coincidence in mathematics can be explained by a common proof, I shall not contend that every proof that constitutes a mathematical explanation renders some theorem non-coincidental or that every explanatory proof derives its explanatory power from the unification it creates. (See n. 10.)

have not misled us — since what they have led us to expect is true: every positive integer possesses this property (Guy 1988, p. 706).

One way to understand the 'misleading fact' suggestion is that a mathematical fact is coincidental exactly when it misleads us. (Sorensen says that from a mathematically omniscient viewpoint, there are no mathematical coincidences, presumably because someone who is mathematically omniscient is not misled.) But this suggestion entails that once we know that $3^4 + 4^4 + 5^4 + 6^4 \neq 7^4$, it is no longer coincidental that $3^2 + 4^2 = 5^2$ and $3^3 + 4^3 + 5^3 = 6^3$ since we are no longer misled into thinking that $3^4 + 4^4 + 5^4 + 6^4 = 7^4$. Yet in mathematical practice, certain facts are still deemed to be mathematical coincidences even after we have stopped being misled by them.

Accordingly, a more promising way to understand the 'misleading fact' suggestion is that a mathematical fact is coincidental in so far as it has the power to mislead those who do not already know that the expectations it tends to provoke (in those who do not already know better) turn out to be false. However, a mathematical fact may still be no coincidence even though it has tremendous capacity to mislead anyone who lacks any better information. For instance, consider this sequence of definite integrals (each running from 0 to $\infty$):

$$\int (1/x) \sin 4x \, \cos x \, dx$$

$$\int (1/x) \sin 4x \, \cos x \, \cos(x/2) \, dx$$

$$\int (1/x) \sin 4x \, \cos x \, \cos(x/2) \cos(x/3) \, dx$$

$$\int (1/x) \sin 4x \, \cos x \, \cos(x/2) \cos(x/3) \cos(x/4) \, dx$$

$$\vdots$$

Suppose we check each of these through

$$\int (1/x) \sin 4x \, \cos x \, \cos(x/2) \, \cos(x/3) \ldots \cos(x/30) \, dx$$

and find that remarkably, each equals $\pi/2$. This result strongly suggests that

$$\int (1/x) \sin 4x \, \cos x \, \cos(x/2) \, \cos(x/3) \ldots \cos(x/31) \, dx = \pi/2.$$

This turns out not to be the case; the sequence is misleading. Yet it is no coincidence that each of the first thirty integrals equals $\pi/2$. We can prove the following theorem (see Lord 2007, p. 283):

> Let $a_1, \ldots, a_n, b$ be positive real numbers. Then $\int (1/x) \sin bx \cos a_1 x \cos a_2 x \ldots \cos a_n x \, dx = \pi/2$ if $a_1 + \ldots + a_n < b$.

Therefore, since $1 + \frac{1}{2} + 1/3 + \ldots + 1/30 < 4$, it is no coincidence that each of the first 30 members of the sequence equals $\pi/2$. ($1 + \frac{1}{2} + 1/3 + \ldots + 1/31 > 4$.)

Thus, a combination of mathematical facts that has great power to mislead those who do not already know better need not be a mathematical coincidence. Conversely, a mathematical coincidence may not be misleading at all. That the thirteenth digits of $\pi$ and $e$ are the same does not mislead anyone into thinking that all of the subsequent digits of $\pi$ and $e$ are the same or that anything else remarkable is going on. (That the thirteenth digits of $\pi$ and $e$ are the same leads Davis (1981, p. 313) to propose and might lead us to think that on average, every tenth digit of $\pi$ and $e$ is the same. But even if this turns out to be true — and so we are not *mis*led — the fact that the thirteenth digits of $\pi$ and $e$ are the same is still a coincidence.) That the formula for the sum of the first $n$ natural numbers is the same, whether $n$ is even or odd, does not immediately suggest any broader result, and so could not have misled us — but for all we initially knew, that the same formula works for both evens and odds might have been coincidental.

Likewise, consider the start of the sequence of numbers $n$ such that $n^4$ contains exactly four copies of each digit in $n$:

> 5702631489, 7264103985, 7602314895, 7824061395, 8105793624, 8174035962, 8304269175, 8904623175, 8923670541, 9451360827, 9785261403, 9804753612, 9846032571, …

For instance, 5702631489 is in the sequence since its fourth power

> 1057550783692741389295697108242363408641

contains four 5s, four 7s, four 0s and so on. All of the terms in the sequence that are ten-digit numbers are pandigital: each contains all of the digits 0 through 9 exactly once. Sloane (2007, sequence A114260) terms this 'probably accidental, but quite curious', and I am inclined to agree. But whether or not this fact turns out to be coincidental, it does not readily lead us to formulate any broader hypotheses. For instance, this fact would not lead us to think that every number in this sequence is pandigital, since numbers longer than ten digits cannot possibly be. (A number $n$ longer than ten digits can belong to the sequence; if such an $n$ contains exactly two 7s, for instance, then $n^4$ contains exactly eight 7s. See Sloan 2007, sequence A114258.)

Furthermore, suppose we saw only the first five numbers $n$ such that $n^4$ contains exactly four copies of each digit in $n$. That these are all pandigital might naturally lead us to expect that any other *ten-digit* number such that $n^4$ contains exactly four copies of each digit in $n$ is likewise pandigital. In forming this expectation, we would not have been *mis*led, since as we have seen, it turns out to be true that all of these ten-digit numbers are pandigital. However, this fact might nevertheless be just a coincidence.

To identify a mathematical coincidence as a misleading fact is to suggest that if a particular mathematical truth leads exclusively to truths when it is generalized in various natural ways, then it must have been no coincidence. But presumably there can be 'giant', 'unrestricted' mathematical coincidences no less than 'little', 'restricted' ones. That a particular mathematical truth leads to truths when it is broadened in a natural way may sometimes be some *evidence* that the particular mathematical truth is no coincidence. But it fails to prove that this is so. (Later I will explain why the fact that a particular mathematical truth leads to truths upon being generalized sometimes counts as evidence that the initial truth is no coincidence.)

Rather than taking a mathematical coincidence to be a fact suggesting further mathematical claims that turn out to be false, we might instead take a mathematical coincidence to be a fact that is misleading in a broader (and vaguer) sense: it does not repay further study, it is not fruitful, it leads to no further interesting mathematics. (This seems to me closest to Sorensen's own view.) The calculator-keyboard fact is then no coincidence because it leads to Nummela's general result regarding numbers having digits extracted from arithmetic sequences, and the summation formulas for even $n$ and odd $n$ are no coincidence because they lead us to another, interesting proof of the same formula for all $n$.

I think that these further results are indeed involved in making the calculator-keyboard fact no coincidence and the summation formulas for even $n$ and odd $n$ no coincidence. But it is not the case that they are non-coincidences *because* they suggest further interesting mathematics. Rather, they suggest further interesting mathematics because they are non-coincidences. It is fruitful to think further about a non-coincidence because we may thereby uncover the facts that make it no coincidence.

The fact that a given mathematical coincidence (such as the coincidence regarding $\pi$ and $e$) leads us to no further interesting mathematics must be sharply distinguished from the fact that its components stand in no interesting mathematical relations to one another. I will suggest that the reason why a genuine mathematical coincidence leads nowhere is that there is nowhere interesting for it to lead: its components lack a certain sort of mathematical relation to one another.

To sum up: epistemic and psychological considerations seem insufficient to reveal what make certain combinations of facts qualify as mathematical coincidences. Even from a mathematically omniscient perspective, there are some mathematical coincidences. Indeed, mathematical omniscience would require knowledge that a given pair of facts forms a mathematical coincidence — that they bear to each other none of the mathematical relations that would make them non-coincidental.

## 3. The components of a mathematical coincidence have no common explanation

Having discovered that every calculator-keyboard number turns out to be divisible by 37, we are inclined to expect there to be some reason why this regularity holds. Likewise, having learned that the two Diophantine equations I mentioned earlier have exactly the same positive solutions, we tend to expect this fact to have some explanation. An attractive suggestion is that the former fact turns out to have an explanation whereas the latter has none — making the latter, but not the former, a mathematical coincidence. In provoking the search for an explanation, the former fact leads to further interesting mathematics but the latter does not. The reason that a coincidence is not mathematically fruitful is because there is no explanation of it to be found.[6]

Of course, this approach replaces the question 'What is a mathematical coincidence?' with another difficult question, 'What is a mathematical explanation?' But even before reaching that question, this approach encounters an obstacle: mathematical coincidences may *have* explanations. For instance, the fact that the Diophantine equations $2x^2(x^2-1) = 3(y^2-1)$ and $x(x-1)/2 = 2^n - 1$ have exactly the same five positive solutions is explained by whatever explains why the positive solutions to the first equation are exactly $x = 1$, 2, 3, 6, and 91, together with the reason why these values are the positive solutions to the second equation. For that matter, to explain why $(n+1)n/2$ gives the sum of the first $n$ natural numbers, whether $n$ is even or odd, it might seem sufficient to give the proof for even $n$ (that proceeds by pairing all $n$ numbers) followed by the proof for odd $n$ (that pairs all numbers except for the middle one). Yet this

[6] Strangely, Davis apparently holds just the opposite view: 'The existence of the coincidence implies the existence of an explanation' (Davis 1981, p. 320).

'explanation' fails to make the result no coincidence. Instead, it prompts us to wonder whether the result is coincidental.

That coincidences have explanations (of a sort) is borne out by non-mathematical examples. For instance, if it is a coincidence that the CIA agent was in the capital just when His Excellency dropped dead, then the explanation of this coincidental fact consists of the causal history of the CIA agent's presence in the capital at that time along with the causal history of His Excellency's death (Lewis 1986b, p. 220).[7]

Of course, this pair of events is coincidental because the two events have no co*mmon* cause — at least, none that is important and of a relevant kind (where importance and relevance depend on the context in which we are discussing the events). For example, the Big Bang is a common cause of the CIA agent's presence and His Excellency's death. Typically, however, the Big Bang is too remote to qualify as a relevant kind of common cause, and so it remains coincidental that the CIA agent was in the capital just when His Excellency dropped dead.

We might try to extend this thought to mathematical coincidences: two (or more) mathematical facts constitute a coincidence when they have no (sufficiently important) common explainer (of a relevant kind).[8] This proposal has the virtue of focusing on the *relation*

[7] It ultimately does not matter to my argument whether we join Hume (1980, p. 56) and say that Lewis's coincidence has an explanation, though there is no single unified explanation of its two components, or whether we join Owens (1992) and say that Lewis's coincidence has no explanation (since there is no single unified explanation of its two components), though each component has an explanation. (Perhaps the right thing to say depends upon the conversational context.) Nothing here turns on whether we credit the conjunction of the two separate explanations of the two components as explaining the coincidence. What is important is that an explanation that identified a common cause of this fact's two components would have supplied a kind of understanding that cannot be supplied by separate treatment of the two components. This point is unaffected by whether separate treatment counts as an explanation of the coincidence. The same point applies to mathematical coincidences. Shortly, I will propose that a mathematical coincidence lacks a certain kind of mathematical explanation (one that, roughly speaking, treats all of its components together). But nothing important differentiates this view from the proposal that a mathematical coincidence lacks any mathematical explanation (because a genuine explanation would have to treat the components together). Whether the mere conjunction of separate explanations of the two components counts as explaining the components' conjunction may depend on whether or not it is coincidental that both components are true. It might be suggested that since it is no coincidence that $(n+1)n/2$ gives the sum of the first $n$ natural numbers for both odd $n$ and even $n$, separate proofs of the summation formula for odd $n$ and for even $n$ fail to correctly explain why it holds for all $n$, since such a pair of separate proofs would incorrectly depict this fact as coincidental. See Sect. 4.

[8] Owens (1992) proposes that a coincidence is any event whose constituents are causally independent (i.e. one does not produce the other and their causal histories have nothing in

between the component facts, a feature that was not emphasized by the earlier proposals (having to do with a single, undecomposed fact's being surprising or leading to no further interesting mathematics, for instance). Clearly, that the sum of the first $n$ natural numbers is $n(n+1)/2$ could be decomposed into many different combinations of facts — for example: (i) that $n(n+1)/2$ gives the sum on Mondays and also on other days of the week, (ii) that $n(n+1)/2$ applies when $n < 100$ and also when $n \geq 100$, and (iii) that $n(n+1)/2$ applies when $n$ is odd and also when $n$ is even. We are never inclined to wonder whether the first pair of facts is a coincidence, nor the second pair, but after seeing the separate proofs (that I gave earlier), one for even $n$ and the other for odd $n$, we might well have found ourselves wondering whether the third pair is a coincidence.

Presumably, then, a given mathematical fact is coincidental (or not) only relative to a particular way of decomposing it. Even if it is coincidental that F and G are both true, and F & G is logically equivalent to H & J, it can be no coincidence that H and J are both true. (It is coincidental that the thirteenth digits of $\pi$ and $e$ are both 9, but it is no coincidence that they are both 9 not only on Mondays, but also on every other day of the week.) Something about the relation between the components determines whether or not it is coincidental that both are true.[9]

However, a coincidence's components may possess a common explainer. After all, many mathematical facts about numbers are presumably explained by the axioms of set theory and logic (along with the definitions of various mathematical concepts in terms of set theory and logic). Accordingly, two such mathematical facts have many

common). Owens does not discuss *mathematical* coincidences; his concern is events and their causal relations. I shall more directly compare mathematical and physical coincidences in Sect. 4.

[9] I presented my four initial examples as undecomposed facts (e.g. that the same summation formula applies to both even $n$ and odd $n$, that $\pi$ and $e$ have the same thirteenth digit). However, especially in the context in which I presented them (e.g. after offering separate derivations for even $n$ and odd $n$), there is a salient way to decompose each of these examples, and we implicitly used that decomposition in considering whether or not these truths are coincidental. Until some decomposition of a given mathematical fact is made salient (by, for instance, a proof that proceeds by cases), there is nothing that its being coincidental (or no coincidence) would amount to. The mere existence of a proof that proceeds by cases and makes salient the theorem's decomposition into those cases does not suffice to make the theorem coincidental. Rather, as I will suggest shortly, there must be no explanation that treats all of those cases together.

explainers in common even though it may remain coincidental that those two mathematical facts hold.

Admittedly, in a given context, the axioms of (and definitions in terms of) set theory and logic might be considered too remote to qualify as a relevant kind of common explainer — just as the Big Bang, although a common cause of the CIA agent's presence and His Excellency's death, is typically considered too remote from these events to keep them from qualifying as coincidental. However, let us suppose that both of the Diophantine equations in the mathematical coincidence we considered earlier could be solved by using the same sophisticated formula applicable to a large class of Diophantine equations (just as the quadratic formula can be used to solve all quadratic equations). Let us also suppose that this formula is salient in the context in which we are examining these two equations. The two derivations using the formula to solve these equations could nevertheless proceed by such dissimilar steps that they converge upon the same expressions only when they have nearly reached their conclusions. That the two equations have the same solutions would then seem like an algebraic miracle. Yet the two components of this coincidence have explanations of the relevant sort that appeal in important respects to the very same fact (namely, the formula).

Perhaps, then, the analogy between mathematical and causal coincidences should be drawn slightly differently: whereas the components of a causal coincidence have no common *causal explainer*, the components of a mathematical coincidence have no common *mathematical explanation*. In particular, there is no single proof by which both are explained. The proof showing it to be no coincidence that the $n(n+1)/2$ formula applies to both odd $n$ and even $n$ is a single proof that covers both of these cases together, unlike the separate proofs for odd $n$ and even $n$ that prompted us to wonder whether or not it was a coincidence that the same formula applies to both. Likewise, Nummela demonstrated the calculator-keyboard fact to be no coincidence by giving a single proof covering all of the calculator-keyboard numbers. In contrast, there is no single proof solving both of the Diophantine equations together or deriving the thirteenth digits of both $\pi$ and $e$. Similarly, that each of the first thirty integrals we saw equals $\pi/2$ is no coincidence (despite being misleading) in virtue of the proof of the general theorem covering those thirty integrals together.

I believe this suggestion to be very much along the right track. The components of a non-coincidence have a unified explanation of a kind that the components of a coincidence lack. Not all proofs (or perhaps even all explanations) of the components of a mathematical non-coincidence succeed in unifying them. But the fact that those components have a unified explanation makes their truth non-coincidental.[10]

To cash out this suggestion, we must take the second proof I gave that the $n(n+1)/2$ summation formula works for all $n$ and we must identify why that proof counts as a *unified* explanation of the formula's holding for both odd $n$ and even $n$—a *common* explanation of the two components. Alternatively, take the separate proofs for even $n$ and odd $n$. Together they prove that the formula holds for all $n$. Why does this conjunction of proofs nevertheless fail to qualify as a 'common' or 'unified' explanation?

Consider the conjunction of the separate proofs of the summation formula for even $n$ and odd $n$. This conjunction has more than is needed to prove that the formula holds for even $n$. When we omit what is not needed, what remains does not suffice to prove that the formula holds for odd $n$. In contrast, take the second proof I gave, which covers all $n$ together. Suppose we insert from the outset the requirement that $n$ be even, so that the argument proves only that the formula holds for even $n$. No part of the argument is dispensable for proving that result. Furthermore, nothing needs to be added to that argument in order to prove that the formula holds for odd $n$ as well. We need only omit the initial restriction to even $n$; no part of the

[10] Kitcher (1976, 1982, 1989) has argued that mathematical explanation should be understood in terms of unification, which he elaborates in terms of many facts being derivable by the same patterns of derivation (argument schemas), thereby reducing the number of types of facts that have to be accepted as brute. Kitcher and I both emphasize common explanatory arguments rather than common explainers (premises of those arguments). But Kitcher is concerned with understanding explanation, not coincidence. He argues that what makes an argument explanatory is its belonging to the best systematization (the one that best satisfies the ideal of allowing the most to be derived by using the fewest patterns of derivation, an ideal reminiscent of Lewis's 'best system' account of natural law). In contrast, I am not attempting to cash out mathematical explanation; I am arguing that what makes for non-coincidence is common explanation. I have not suggested that all explanations unify by reducing the number of separate types of argument in the systematization. (Tappenden 2005 is a useful critique of Kitcher's account of mathematical explanation.) Moreover, Kitcher appeals to the notion of an argument schema (since unification is produced by the repeated use of arguments having the same schema), leaving him vulnerable to worries (he tries to address) about what should count as gerrymandering a given schema's boundaries. I do not appeal to the notion of a schema; on my proposal, every component of a mathematical non-coincidence can be proved by the same argument (not by arguments having the same schema). Finally, Kitcher (1975, p. 265) regards proofs by mathematical induction as explanatory, whereas shortly I argue that generally they are not (even though their argument schema allows a great deal to be derived).

argument depended on that restriction, so once it is omitted, nothing needs to be added to yield the result for all *n*, odd and even.

The same approach accounts for the way that Nummela's demonstration makes the calculator-keyboard result qualify as no coincidence. We can prove that result by taking each of the sixteen calculator-keyboard numbers, individually showing that number to be divisible by 37, and then conjoining the sixteen proofs. If we omit from this proof whatever is unnecessary for showing that (for instance) 123321 is divisible by 37, then we omit the treatment of the other fifteen numbers. What remains cannot show that (say) 321123 is divisible by 37. On the other hand, suppose that we take Nummela's argument and use it to show that 123321 is divisible by 37. That argument begins by noting that 123321 takes the form

$$a.10^5 + (a + d).10^4 + (a + 2d).10^3 + (a + 2d).10^2 + (a + d).10 + a.1$$

where $a$, $a + d$, and $a + 2d$ are three integers in arithmetic progression. To extend this argument to show that every calculator-keyboard number is divisible by 37, we need only omit the initial restriction to 123321. Nothing needs to be added to the argument's other steps in order to cover the other fifteen calculator-keyboard numbers since all of them take the above form.

No such unified proof exists of the fact that the Diophantine equations $2x^2(x^2 - 1) = 3(y^2 - 1)$ and $x(x - 1)/2 = 2^n - 1$ have exactly the same five positive solutions. (Or, at least, so I presume, in presuming this combination of facts to be coincidental.) We could take separate procedures for solving the two equations and cobble them together into one proof. But the steps of the procedure solving one equation could then be omitted without keeping the proof from solving the other equation. The stripped-down proof would be unable to solve the first equation.

In short, then, the components of a non-coincidence can all be given the same explanation in that there is a proof explaining them all that possesses the following feature in connection with at least one of the components. Suppose we take that single component of the non-coincidence and make each step of the proof as logically weak as it can afford to be while still allowing the proof to explain that component. Then the weakened proof remains able to explain each of the non-coincidence's other components as well (once we remove any restrictions to that single component, such as 'Let *n* be even' for the summation formula). No further resources are needed to expand the explanation's scope to cover all components of the non-coincidence. In contrast, the various components of a mathematical coincidence lack a single, unified explanation of this kind. Thus, for mathematical truths F, G, H …

> It is a coincidence that F and G and H … are all true if and only if F, G, H … have no single, unified explanation

Admittedly, this elaboration of a 'single, unified explanation' may be vague at the margins — for instance, in whether a given proof explaining one component can be expanded to cover another component merely by removing an otiose restriction, or only by adding some slight further resource. But our notion of a 'mathematical coincidence' will, I suspect, be correspondingly vague in marginal cases. What count as 'further resources' may also be context sensitive. That F and G both hold may qualify in a given context as no coincidence if the only further resource needed by some proof explaining F in order also to explain G is regarded in that context as negligible. (We will see an example in the following section.)

Here is another kind of intermediate case: Consider

(A) The number of points in a plane that uniquely determine a conic section

(B) The degree of the alternating group that is the smallest non-Abelian simple group

(C) The smallest degree general algebraic equation that is not solvable in closed form

(D) The smallest $n$ such that every $n^{\text{th}}$ Fibonacci number gains another digit in its decimal expansion

It turns out (I believe) to be no coincidence that the same number (namely, 5) is both (B) and (C). However, for each other pair of the above, it is a coincidence that they refer to the same number (again, 5). Now what about the fact that (A), (B), and (C) all pick out the same number? By my account, it is a coincidence, since its three components have no single, unified explanation. However, I think it would sometimes be misleading to characterize this fact as coincidental and simply leave it at that, since it is no coincidence that (B) and (C) denote the same number. One might be inclined to say that it is not a *complete* coincidence that (A), (B), and (C) all refer to the same number. Although it is a coincidence, strictly speaking, there is a sense in which it is not *as much* of a coincidence as the fact that (A), (B), and (D) all denote the same number. My account can accommodate such 'degrees' of being a coincidence.[11]

My approach also captures the grain of truth in the idea that if it is a coincidence that F and G are both true, then G's truth remains surprising even in light of F's truth: an explanation of F that is no stronger than it needs to be fails to explain G. Furthermore, this approach entails that 333 333 331's character as composite (i.e. not a prime number – as the product of 17 and 19 607 843), though insufficient to make it just a coincidence that 31, 331, 3 331, … , and 33 333 331 are all prime, is nevertheless relevant to its being coincidental. That 333 333 331 is not prime precludes a general theorem that any number is prime if every digit in its decimal representation is '3' except for a

[11] Another kind of intermediate case: a theorem that has no single, unified explanation, but although one explanation of the theorem decomposes it as F & G, another explanatory proof of the theorem cross-cuts these components. (That is, there is another explanation that decomposes the theorem as H & J where H and J each includes some of the cases covered by F and some covered by G.) The components F and G are not fully unified, but neither must they receive entirely separate treatment.

A special case: suppose that one component is a mathematical axiom. Perhaps an axiom has no explanation. Then the various components lack a single, unified explanation of the kind I have just described, since one of the components has no explanation at all! But the fact that all of those components are true is not then obviously coincidental — especially in a case where the axiom explains the other component's truth. One component's lack of an explanation should not suffice to make it coincidental that all of the components are true. (On the other hand, perhaps axioms explain themselves, in which case they present no problem for my proposal.). The only sorts of cases that I have seen characterized as 'coincidental' (or not) in mathematical practice are cases where none of the components is an axiom. Consequently, the proper way to characterize cases where one component has no explanation seems to me a decidedly peripheral matter worthy of being treated as a special case. One option would be to restrict the necessary and sufficient conditions for mathematical coincidence that I give in the main text so that they apply only if every component has an explanation. The account could then be extended as follows to cover cases where some component has no explanation: It is a coincidence that F and G and H … are all true if and only if the components having explanations have no common explanation (in the sense I describe in the main text) or a component having no explanation figures in no such common explanation. Thus, if F is an axiom and has no explanation, whereas G is explained by F, then it is no coincidence that F and G both hold, since G's explanation by F is trivially a common explanation of all of the components having explanations, and F figures in that explanation. (This proposal entails that if F and G are independent axioms without explanations, then it is coincidental that F and G are both true. Again, in view of mathematical practice, how we characterize such a case seems to me of little interest.)

My view allows it to be a coincidence that F and G are both true, even if it is no coincidence that F and (F and G) are both true because the conjunction of the separate explanations of F and G counts as an explanation of (F and G) that needs no additional resources to explain F. As we saw earlier, it can be a coincidence that F and G are both true, even if F & G is logically equivalent to H & J and it is no coincidence that H and J are both true.

final '1'—and so forecloses a unified explanation of such a theorem, which would make it no coincidence that 31, 331, 3 331, … , and 33 333 331 are all prime.[12] Likewise, before you knew that 333 333 331 is not prime, you might well have taken the fact that 33 331, 333 331, 3 333 331, and 33 333 331 are all prime as some evidence that it is no coincidence that 31, 331, and 3 331 are all prime, since it was some evidence that there is a general theorem (having a unified explanation) that all numbers of this form are prime—which would make it no coincidence.

To make various components non-coincidental, their common proof must be an *explanation*, not merely a *deduction*.[13] For example, consider the following example of a mathematical coincidence:

> [C]onsider the decimal expansion of *e*, which begins 2.718281828 … It is quite striking that a pattern of four digits should repeat itself so soon—if you choose a random sequence of digits then the chances of such a pattern appearing would be one in several thousand—and yet this phenomenon is universally regarded as an amusing coincidence … (Gowers 2007, p. 34)

Of course, there are many ways to derive *e*'s value, and each of them consists of a common proof of the third-through-sixth digits together with the seventh-through-tenth digits. For example, we could derive them from the fact that *e* equals the sum of $(1/n!)$ for $n = 0, 1, 2, 3, \ldots$. However, such a common proof does not *explain why* the seventh-through-tenth digits are 1828 or why they repeat the third-through-sixth digits. It merely proves that they are and do. There is, I suggest, no reason why this pattern of digits repeats. It just does.[14] (Of course,

[12] Leavitt (2007, p. 182) characterizes the fact that these are all prime as a 'coincidence', considering that 333 333 331 is not prime.

[13] If we take proofs of two arbitrary theorems and combine their steps by using gerrymandered, wildly disjunctive, gruesome predicates, then the resulting proof is typically not an explanation of the resulting theorem. Its predicates do not refer to mathematical natural properties and kinds. An explanation of some wildly disjunctive theorem 'All triangles or prime numbers are … ' consists of an explanation of the triangle result together with an explanation of the result concerning prime numbers. (For more on mathematical natural properties and kinds, see Corfield 2005 and Tappenden 2008.)

[14] By the same token, there might be a proof that the thirteenth digits of $\pi$ and $e$ are the same that does not proceed by first computing $\pi$ to thirteen digits and then computing $e$ to thirteen digits. Instead, the proof might deduce various independent linear combinations of $\pi$ and $e$ (e.g. $(\pi + e)/2$, $2\pi + 3e$), from which $\pi - e$, and ultimately perhaps even the thirteenth digits themselves, might be inferred. (That the thirteenth digit of $\pi - e$ is 0 would show that $\pi$'s and $e$'s thirteenth digits differ by 1 at most.) After all, as Tim Gowers suggested to me in pressing this point, there are ways of calculating linear combinations of $\sqrt{2}$ and $\sqrt{3}$ without calculating either $\sqrt{2}$ or $\sqrt{3}$. Would such a proof—every part of which concerns both $\pi$ and $e$—give the components of this coincidence a 'single, unified explanation'? Not if the proof fails to *explain why* those linear combinations take on their values.

my account of mathematical coincidence leaves us with the task of understanding what a mathematical explanation is, and of doing so without appealing to the notion of a mathematical coincidence. Fair enough — but my account of mathematical coincidence is not thereby rendered empty or uninteresting.)

Here is another example. We could deduce that the sum $S$ of the first $n$ natural numbers is $n(n+1)/2$ from the premise that $S^2 = (n^4 + 2n^3 + n^2)/4$, with the same proof whether $n$ is even or odd. But this common proof would do nothing to rule out the possibility that the same formula holds for even $n$ and odd $n$ only as a matter of coincidence. Presumably, the fact that $S^2 = (n^4 + 2n^3 + n^2)/4$ does not explain why $S = n(n+1)/2$. If anything, the expression for $S$ explains the expression for $S^2$.

Likewise, consider the proof of the summation formula that proceeds by mathematical induction:

> Show that for any natural number $n$, the sum of the first $n$ natural numbers is equal to $n(n+1)/2$.
>
> For $n=1$, the sum is 1, and $n(n+1)/2 = 1(2)/2 = 1$.
>
> If the summation formula is correct for $n=k$, then the sum of the first $k+1$ natural numbers is $[k(k+1)/2] + (k+1) = (k+1)[(k/2)+1] = (k+1)(k+2)/2$, so the summation formula is correct for $n = k+1$.

For this argument to be explanatory, the fact that the summation formula works for every natural number would have to be explained in part by the fact that the summation formula works for $n=1$. But the case of $n=1$, though more mathematically tractable than other cases, seems to have no special explanatory privilege over them.

After all, we could just as well have started our proof with $n=5$, working upward and downward from there:

> For $n=5$, the sum is $1+2+3+4+5=15$, and $n(n+1)/2 = 5(6)/2 = 15$.
>
> If the summation formula is correct for $n=k$, then (I showed earlier) it is correct for $n=k+1$.
>
> If the summation formula is correct for $n=k$ (where $k>1$), then the sum of the first $k-1$ natural numbers is $[k(k+1)/2] - k = k[(k+1)/2 - 1] = k(k-1)/2$, so the summation formula is correct for $n=k-1$.

If the proof by mathematical induction is explanatory, then this very similar proof proceeding upwards and downwards from $n = 5$ should presumably also be explanatory. There is nothing to distinguish the two proofs, except for where they start. But they cannot *both* be explanatory, since then the summation formula's working for $n = 1$ would help to explain why it works for $n = 5$ *and* the summation formula's working for $n = 5$ would help to explain why it works for $n = 1$. Mathematical explanations, whatever they are, had better not run in a circle.[15]

Thus, the proof by mathematical induction, though treating even $n$ and odd $n$ together, fails to make it no coincidence that $n(n+1)/2$ applies to both even $n$ and odd $n$ because it fails to explain *why* the

[15] If mathematical 'explanations' run in a circle, then they are nothing like scientific explanations; the mathematical explanans is not *responsible* for the explanandum. Of course, this argument might lead us to worry that there is no such thing as mathematical 'explanation' properly so called, since there are many, equally good ways to axiomatize mathematics, and under different axiomatizations, the 'explanatory' arguments run in opposite directions. Any account of what a mathematical explanation is must face this problem, and I offer no such account here. I am using the notion of a mathematical explanation to understand the phenomenon of mathematical coincidence, and in so far as this notion is useful in understanding mathematical coincidence and other phenomena, there is good reason to suppose that there is such a thing as mathematical explanation. (See Sect. 4.) For more comprehensive arguments that mathematical explanation is an important part of mathematical practice, along with many examples, see Hafner and Mancuso 2005 and references therein. (See also note 4.) I shall resist the urge to say anything more about mathematical explanation except this: That a proof's explanatory power depends somehow on its audience's interests, broadly speaking, does not entail that a proof's explanatory power for some audience is nothing more than its being the kind of proof that the audience wants for whatever reason (e.g. in view of its premises, its strategy, its perspicuity, its brevity, its wit, or the collateral information it supplies). (Contrast Resnik and Kushner 1987.)

My argument that mathematical inductions are generally not explanatory (see Lange 2009a) presupposes that if the fact that the summation formula works for $n = 1$ helps to explain why it works for every natural number, then in particular, the fact that the summation formula works for $n = 1$ must help to explain why it works for $n = 5$. I do not base this step on the premise that if a fact helps to explain a given universal generalization, then it must help to explain every instance of that generalization. For instance, various past decisions that my wife and I made help to explain why we have exactly two children, and thereby help to explain why every family on my block has exactly two children (a coincidence), but they do not help to explain why our neighbors, the Smith's, have exactly two children. Rather, certain facts about my family's history explain our case, whereas various facts about the Smith's history explain their case. However, this kind of piecemeal explanation would not be taking place if the principle of mathematical induction together with the summation formula's working for $n = 1$ helped to explain why it works for every natural number; it would not be the case that the principle of induction sufficed to explain why the summation formula works for $n = 5$ whereas the formula's working for $n = 1$ sufficed to explain why it works for (say) $n = 12$. (Thanks to Alexander Skiles for discussion of this point.)

formula holds; it merely shows *that* the formula holds.[16] However, suppose we take the argument that proceeds upwards and downwards from $n = 5$ and omit the step showing that the summation formula works for $n = 5$. Instead, we work upwards and downwards from an arbitrary $n$ to show that if the summation formula works for one natural number $n$, then it works for all others. Since this argument fails to privilege any value of $n$, it may well count as explaining why the summation formula works either for all $n$ or for no $n$—that is, as explaining why it is not the case that the summation formula works for some but not all $n$. Furthermore, this argument is unified in its treatment of odd $n$ and even $n$ in the manner I described earlier. After all, for the argument to begin from some $n$ and to proceed upwards and downwards by steps of 1 each time, ultimately to cover every odd $n$, the argument must along the way cover every even $n$ as well. Given that the summation formula works for all $n$, this explanation of why it works either for all or for no $n$ ties together the even component and the odd component so as to make it no coincidence that the formula applies both to even $n$ and odd $n$.

Therefore, I must weaken slightly the necessary and sufficient conditions I just proposed for some combination of mathematical facts to be no coincidence. Those conditions demanded a common proof for all components. But now we see that this proof need not explain why (or even prove that) every component of the non-coincidence is true. It need only explain why (and prove that) all of the components of the non-coincidence are true *if any one is true*—that is, why they all stand or fall together. As before, each component must receive the same explanation: if we take a single component and make each step of

[16] Hafner and Mancosu (2005, p. 237) and Hanna (1990, pp. 10–11) contend that according to working mathematicians, proofs by mathematical induction are paradigms of non-explanatory proofs—but they offer no account of why proofs by mathematical induction fail to explain.

Some philosophers may believe that 1 is ontologically prior to the other natural numbers—for instance, that the natural numbers are fundamentally an inductively generated set with 1 as its first element. Whatever argument these philosophers give that 1 is prior would break the symmetry between a proof of the summation formula by mathematical induction and its proof by proceeding upwards and downwards from 5. I have not contended that all mathematical inductions fail to explain or that the summation formula's inductive proof fails to explain—merely that it explains only if the formula's working for $n = 1$ is somehow explanatorily prior to its working for $n = 5$. My main concern here will shortly be to explain why we must weaken slightly the conditions that I have just proposed for some combination of mathematical facts to be no coincidence. For that purpose, it suffices that we accept at least for the sake of argument that the inductive proof of the summation formula is not explanatory.

the proof as logically weak as it can afford to be, while still allowing the proof to prove that that component is true if any one of the components is true, then the weakened proof must be able to prove, for each of the other components as well, that it is true if any one of the components is true.[17]

## 4. Two applications: mathematical and scientific explanation

I have suggested that there is a close connection between mathematical explanation and mathematical coincidence: two mathematical facts are no coincidence when they have a common mathematical explanation. Of course, there is some controversy in philosophy over whether or not there are any explanations in mathematics. I suggest that any motivation we may have for thinking that there are mathematical coincidences should also motivate us to think that there are mathematical explanations, since the notion of a mathematical coincidence can be understood only in terms of the notion of a mathematical explanation.

Here is an example that nicely brings out the connection between mathematical explanation and mathematical coincidence.

[17] According to my proposal, it is a coincidence that the thirteenth digit of $\pi$ is the same as the thirteenth digit of *e*. By the same token, my proposal entails that it is a coincidence that (say) the twelfth digit of $\pi$ is 8 and the fifth digit of *e* is 2. It might seem that these two cases ought to be treated differently. In particular, it might appear that my proposal fails to account for the strangeness of even asking whether it is a coincidence that the twelfth digit of $\pi$ is 8 and the fifth digit of *e* is 2. However, I believe the strangeness to be adequately accounted for by our having no reason at all to suspect that these two facts have a common explanation. That $\pi$ and *e* have the same digit (9) in the same decimal place (thirteenth) might have suggested that there was a common explanation, but that they have different digits in different places is not at all suggestive. Analogous considerations apply to causal coincidences. That you and I are both at the mall this afternoon is (let us suppose) coincidental, and although it is no less coincidental that you are there and Honolulu is the capital of Hawaii, to call this fact 'coincidental' (or even to ask whether or not it is) may well mislead by suggesting that one might reasonably have suspected a certain kind of common cause.

In addition, though, it may be that the distinction between proofs that explain and proofs that merely prove (and hence the distinction between mathematical coincidences and non-coincidences) arises only in a context where some feature of the result being proved is salient. If so, then to characterize as 'coincidental' the fact that the twelfth digit of $\pi$ is 8 and the fifth digit of *e* is 2 is to presuppose that this fact exhibits some salient feature — when actually it may well not. Without some salient feature, it may not make sense even to ask whether or not it is a coincidence. (See also note 19.)

Consider the fraction $1/(1 + x^2)$. By long division,

$$
\begin{array}{r l}
 & 1 - x^2 + x^4 - \ldots \\
(1 + x^2)\,\big) & 1 \\
 & \underline{-(1 + x^2)} \\
 & \quad - x^2 \\
 & \quad \underline{-(-x^2 - x^4)} \\
 & \qquad\qquad x^4 \\
 & \qquad\qquad \underline{-(x^4 + x^6)} \\
 & \qquad\qquad\quad \vdots
\end{array}
$$

it yields the Taylor series

$$1 - x^2 + x^4 - x^6 + \ldots.$$

Plainly, for real number $x$, this series will converge only if $|x| < 1$. (When $|x| > 1$, each successive term's absolute value is greater than its predecessor's, so the sum will oscillate in an ever widening manner.)

Why does dividing $(1 + x^2)$ into 1 generate a series that converges if $|x| < 1$ but diverges if $|x| > 1$? After all, $1/(1 + x^2)$ is perfectly well-defined for $|x| \geq 1$. We have here a mathematical why-question that seems to demand an answer: a mathematical explanation. We can prove that the series converges only if $|x| < 1$, but such a proof may still leave the why-question unanswered. As Michael Spivak says in his famous textbook:

> A careful assessment of our situation will reveal some unexplained facts. … [M]ysterious behavior is exemplified … strikingly by the function $f(x) = 1/(1 + x^2)$, an infinitely differentiable function which is the next best thing to a polynomial function. … If $|x| \geq 1$, the Taylor series does not converge at all. Why? What unseen obstacle prevents the Taylor series from extending past 1 and –1? Asking this sort of question is always dangerous, since we may have to settle for an unsympathetic answer: it happens because it happens — that's the way things are! In this case there does happen to be an explanation, but this explanation is impossible to give [here at the end of Chapter 23]; although the question is about real numbers, it can be answered intelligently only when placed in a broader context. (Spivak 1980, p. 482)

Those who are suspicious of the very concept of a mathematical explanation will need to explain away remarks like Spivak's.

The mathematical explanation that Spivak foreshadows is given by the proof of this theorem (see Spivak 1980, p. 524):

> For any power series $\sum a_n z^n$ (from $n = 0$ to $\infty$), either it converges for all complex numbers $z$, or it converges only for $z = 0$, or there is a number $R > 0$ such that it converges if $|z| < R$ and diverges if $|z| > R$.

This theorem's proof 'helps explain the behavior of certain Taylor series obtained for real functions, and gives the promised answers to the questions raised at the end of Chapter 23' (Spivak 1980, p. 528). For although $f(z) = 1/(1 + z^2)$ does nothing outrageous when $z$ equals a real number with absolute value equal to 1 (remaining well-defined, continuous, and infinitely differentiable), it does go undefined for an *imaginary* number with absolute value (or 'modulus') equal to 1: $f(i) = 1/(1 + i^2) = 1/0$. That is why the Taylor series diverges at $x > 1$.

We can recognize that this is indeed a genuine mathematical explanation by noticing that this theorem's proof removes what would otherwise have been a mathematical coincidence. Is it a coincidence that the two Taylor series

$$1/(1 - x^2) = 1 + x^2 + x^4 + x^6 + \ldots$$

$$1/(1 + x^2) = 1 - x^2 + x^4 - x^6 + \ldots$$

are alike in that, for real $x$, each converges when $|x| < 1$ but diverges when $|x| > 1$? This might seem utterly coincidental (like the fact that the two Diophantine equations have exactly the same positive solutions) since although $1/(1 - x^2)$ at $x = 1$ goes undefined, $1/(1 + x^2)$ behaves quite soberly there. However, when we look at matters in terms of the complex plane, it is no coincidence: both functions go undefined at some point on the unit circle centered at the origin (the first at $z = 1$, the second at $z = i$), so by the theorem, each series converges inside the circle but not outside it. The behaviors of the two series have a common mathematical explanation given by the proof of this theorem. That explanation makes their similar behavior no coincidence.[18]

[18] Steiner (1978b, pp. 18–19; 1990, pp. 105–7) likewise says that this theorem concerning complex power series explains why the Taylor series for $1/(1 + x^2)$ converges at $|x| < 1$ but not at $|x| > 1$ — and, as Steiner notes (1990, p. 106), Waismann says the same (in his 1982, pp. 29–30). Neither mentions mathematical coincidence or compares the convergence of the series for

(Admittedly, to explain the behavior of the $1/(1 + x^2)$ series, a proof explaining the theorem must be supplemented by the fact that $1/(1 + z^2)$ goes undefined at $z = i$. This fact, in turn, plays no role in explaining the behavior of the $1/(1 - x^2)$ series; besides the theorem's proof, that explanation also appeals to the fact that $1/(1 - z^2)$ goes undefined at $z = 1$. However, in a context where infinite series and the like are being placed on the table, we ordinarily would regard these humble further resources as quite negligible. In such a context, a proof explaining the theorem qualifies as a single, unified explanation of the behaviors of both series, making their similarity no coincidence.)

The force of a question like 'Is it just a coincidence that the Taylor series for $1/(1 + x^2)$ and $1/(1 - x^2)$ both converge at $|x| < 1$ but not at $|x| > 1$?' may well be easier to grasp than the force of a question like 'What explains why the Taylor series for $1/(1 + x^2)$ converges at $|x| < 1$ but not at $|x| > 1$?' It may initially be difficult to see what could possibly constitute such an explanation, whereas it is easy to appreciate the 'Is it just a coincidence … ' question as seeking some feature common to the two functions that makes their Taylor series behave alike despite the functions' profoundly different behaviors at $x = 1$. A similar phenomenon occurs in connection with scientific explanation: Encountering the question 'Why is gravity an inverse-square force?', a student may well find it difficult to see what this question could possibly be looking for. But she may readily understand the point of asking 'Is it nothing but a coincidence that both gravity and electrostatic repulsion are inverse-square forces?'

Thus, in a given case, mathematical coincidence may be easier than mathematical explanation to recognize as an interesting issue. Although one might initially think that there is no sense in asking for a mathematical explanation over and above a proof of the non-convergence of $1/(1 + x^2)$'s series at $|x| > 1$, one might nevertheless promptly recognize that there is a fact (albeit perhaps unknown) about whether the common behavior of the two series is a mathematical coincidence. In this light, one might then recognize a distinction

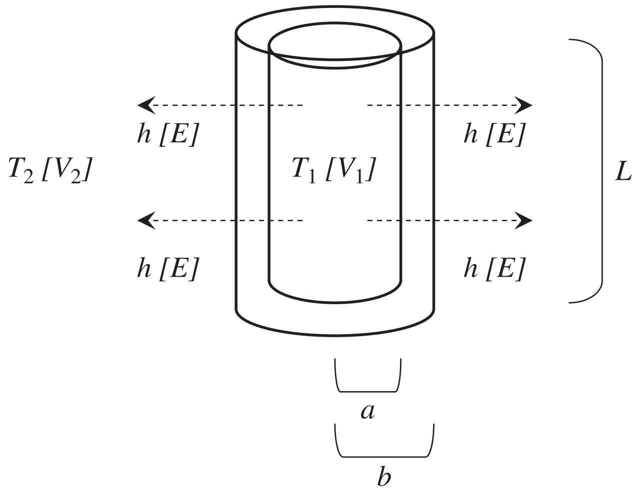
$1/(1 - x^2)$, though both are concerned with the remarkable fact (a coincidence?) that imaginary numbers arise in apparently unrelated branches of mathematics (e.g. the factorization of polynomials and the convergence of power series). (My thanks to a referee for kindly calling Steiner's papers to my attention.)

between proofs that explain and proofs that fail to explain why the $1/(1+x^2)$ series is non-convergent for $|x|>1$.[19]

Even if (as I have argued) a proof that makes some result non-coincidental does so by virtue of being explanatory, we may come to recognize a given proof as explanatory by first appreciating that it makes some result non-coincidental. Indeed, someone who initially doubts that there is any such thing as a 'mathematical explanation' may nevertheless readily appreciate some proofs as rendering non-coincidental what would otherwise be counted as mathematical coincidences — and may thereby be persuaded that there are mathematical explanations.

Besides giving us some purchase on *mathematical* explanations, my account of mathematical coincidence can also help us to understand a certain kind of *scientific* explanation. As an example, let us take these two phenomenological regularities:




Consider a cylinder (see figure) of length $L$, radius $a$, generating heat that keeps it at constant temperature $T_1$. (It might be a current-carrying wire or a steam-conveying airduct, for example.) The cylinder is surrounded by a uniform layer of thermal insulation, thickness $(b-a)$, the outside of which is kept at temperature $T_2$. We find experimentally that in all such cases, the rate at which heat is generated inside the cylinder, thence to pass through

[19] Nevertheless, perhaps this distinction exists only in certain conversational contexts, such as a context where it is salient that some other series exhibits the same convergence behavior. Perhaps in such a context, an explanation of one series' convergence behavior would have to exploit its possession of a property possessed by the other series as well, thereby explaining why they both exhibit this behavior. (See also n. 17.)

> the cylinder's surface, is proportional to $L\ (T_1 - T_2)/\ln(b/a)$, where '$\ln(x)$' is $x$'s natural logarithm. (Arrows in the figure depict the direction of heat flow $h$.)
>
> Consider a cylinder (see figure again, now using the labels in square brackets) of electrically conductive material of length $L$, radius $a$, held at a constant voltage (electrical potential) $V_1$. The cylinder is surrounded by a uniform layer of electrical insulation (dielectric), thickness $(b - a)$, the outside of which is kept at voltage $V_2$. We find experimentally that in all such cases, the charge on the cylinder is proportional to $L\ (V_1 - V_2)/\ln(b/a)$. (Arrows in the figure depict the direction of the electric field $E$.)

Is it a coincidence that these two results are so strongly analogous? Yes, in one respect: the two physical processes are clearly distinct. They cannot be reduced to a common underlying physical process in the way that the tides, planetary motion, and falling bodies can. After all, the electric field is not heat. The laws of electrostatics do not govern thermodynamics.

But in another respect, it is *no* coincidence that the two results are analogous. The analogy would have been coincidental if the more fundamental laws of electrostatics and thermodynamics had been utterly unlike. But in fact, they take exactly the same form, and that is why these two results take the same form. Of course, if we gerrymander or are excessively liberal about what counts as a 'form', then any two equations take the 'same form'. But the laws governing electrostatics and thermodynamics take the same form in the sense that there is a single mathematical proof deriving both of the phenomenological results I mentioned from the respective more fundamental laws.

Let us see why. There is what James Clerk Maxwell called a 'physical analogy' between electrostatics and heat flow: a positively (negatively) charged body plays the nomic role of a heat source (sink), an electric field plays the role of an unequally heated body, potential difference plays the role of temperature difference, and so forth. Since analogous quantities play analogous nomic roles, 'any result we may have obtained either about electricity or about the conduction of heat may be at once translated out of the language of the one science into that of the other without fear of error' (Maxwell 1881, p. 52). That translatability reflects the fact that the same mathematical derivation from the respective more fundamental laws covers both phenomenological results; each step in the derivation can equally well involve electrostatic or thermodynamic quantities. This physical analogy has the practical advantage of giving us two results for the price of one derivation. More importantly for our purposes, the physical analogy makes it no *mathematical* coincidence that these two phenomenological results are so similar.

The relevant more fundamental equations of electrostatics [thermodynamics] are as follows: (i) that the rate of heat flow $\boldsymbol{h}$ [the electric field $\boldsymbol{E}$] at a given point is proportional to the negation of the temperature $T$ [electrical potential $V$] gradient there, and (ii) that the heat flux $\int \boldsymbol{h} \cdot \boldsymbol{ds}$ [electric field flux $\int \boldsymbol{E} \cdot \boldsymbol{ds}$] through an enclosing surface is proportional to the heat $Q$ generated [electric charge $q$] within. That is (where '$\propto$' represents 'is proportional to'):

$$\boldsymbol{h} \propto -\mathbf{grad}\ T \qquad \boldsymbol{E} \propto -\mathbf{grad}\ V$$
$$\int \boldsymbol{h} \cdot \boldsymbol{ds} \propto Q \qquad \int \boldsymbol{E} \cdot \boldsymbol{ds} \propto q.$$

Note the analogy between the two sets of equations. The two phenomenological results can thus be given the same derivation. (I shall use the thermodynamic variables.)

> From symmetry, $\boldsymbol{h}$ is independent of the direction, so it depends only on the distance from the apparatus' center line. Consider a cylinder concentric with the apparatus, with radius $r$. Its surface area is $2\pi rL$. For this surface, that $\int \boldsymbol{h} \cdot \boldsymbol{ds} \propto Q$ entails that $2\pi rLh \propto Q$, and so $h \propto Q/2\pi rL$. Since $\boldsymbol{h} \propto -\mathbf{grad}\ T$, it follows that $h \propto -\mathrm{d}T/\mathrm{d}r$, or $\mathrm{d}T/\mathrm{d}r \propto -h$. Substituting for $h$, $\mathrm{d}T/\mathrm{d}r \propto -Q/2\pi rL$. Integrating from $r = a$ to $r = b$, we find $\int(\mathrm{d}T/\mathrm{d}r)\ \mathrm{d}r = (T_2 - T_1) \propto -(Q/2\pi L)\ \ln(b/a)$. Thus, $Q \propto L\ (T_1 - T_2)/\ln(b/a)$.

If we exchange thermodynamic quantities for electrostatic ones, we can use the same mathematical proof to yield the electrostatic result (Feynman, Leighton, and Sands 1963, p. 12-2).[20]

Thus, considering the analogy between the more fundamental equations of thermodynamics and electrostatics, it is no coincidence that the phenomenological thermodynamic and electrostatic results are analogous. That is to say, this combination of thermodynamic and electrostatic results is no *mathematical* coincidence: a mathematical proof that is no stronger than it needs to be in order to derive the thermodynamic result from the more fundamental thermodynamic equations also suffices to derive the electrostatic result from the more fundamental electrostatic equations.

Of course, since heat and electric charge are not the same thing, there is no scientific explanation of the more fundamental

[20] An analogous point could be made regarding the parallel proofs of dual theorems in projective geometry.

thermodynamic equations that appeals to exactly the same facts (or even nearly so) as some explanation of the more fundamental electrostatic equations. Plausibly, when we say that it is a coincidence that the CIA agent was in the capital just when His Excellency dropped dead, we mean that these two facts have no causal explanations (of a relevant sort) that appeal (in important respects) to the same facts. Such a coincidence, involving the absence of common scientific explainers, I will term a '*physical* coincidence' to distinguish it from a *mathematical* coincidence, which (I have suggested) involves the absence of a common mathematical explanation.

These two notions are similar enough that it makes sense for us to regard both as varieties of coincidence. Yet they are distinct, as the thermodynamic/electrostatic case illustrates. That the phenomenological thermodynamic and electrostatic results are analogous is a *physical* coincidence, since the more fundamental laws responsible for each result are not the same.[21] But it is no *mathematical* coincidence, since the same mathematical proof can be used to derive each phenomenological result from the more fundamental equations that scientifically explain it.

The analogy between the phenomenological thermodynamic and electrostatic results can be *physically* coincidental yet *mathematically* no coincidence because the more fundamental equations responsible for the respective results concern different things (heat and electric charge) yet have a common mathematical form. From mathematical equations of that form, we can derive a mathematical result — and the variables in those equations and in that result can be interpreted either thermodynamically or electrostatically. So there is a mathematical derivation (given above) of the thermodynamic regularity from the more fundamental thermodynamic equations such that (even if we make each step of the derivation as logically weak as it can afford to be while still allowing the proof to go through) the same proof can derive the electrostatic regularity from the more fundamental electrostatic

[21] Though physically coincidental, the fact that the phenomenological thermodynamic and electrostatic results are analogous is physically necessary (i.e. follows entirely from natural laws). That two facts are physically necessary does not suffice to make a given consequence of them no physical coincidence. For instance, nineteenth-century chemists believed it physically necessary that all noncyclic alkane hydrocarbons differ in molecular weight by integral multiples of 14 units, and they also believed it physically necessary that the atomic weight of nitrogen is 14 units. But they termed it 'coincidental' (albeit physically necessary) that all noncyclic alkanes differ in molecular weight by integral multiples of the atomic weight of nitrogen. Noncyclic alkanes contain no nitrogen. (See, for instance, van Spronsen 1969, pp. 73–4, and my 2000, pp. 203–7.)

equations. In other words, the two physical theories leave us with exactly the same *mathematical* problem, and therefore it is no *mathematical* coincidence that the two problems have analogous answers. William Thomson (later Lord Kelvin), who discovered this analogy, put the point nicely:

> Corresponding to every problem relative to the distribution of electricity on conductors, or to forces of attraction or repulsion exercised by electrified bodies, there is a problem in the uniform motion of heat which presents the same analytical conditions, and *which, therefore, considered mathematically, is the same problem.* (Thomson 1845, p. 27, my emphasis)

Without the notion of a *mathematical* coincidence, we would be unable to specify why a correct scientific explanation of the similarity between the two phenomenological results cannot be simply a scientific explanation of the first result conjoined to a scientific explanation of the second. Such a conjunction might seem perfectly appropriate as an explanation considering that their similarity is a *physical* coincidence. But it incorrectly characterizes their similarity as a *mathematical* coincidence — as just a kind of algebraic miracle. In contrast, any correct scientific explanation of their similarity identifies the particular mathematical features that, by being present in both cases, account for the similarity between the two results by enabling the same mathematical argument to derive those results from the respective more fundamental equations.[22]

[22] Colyvan has similarly suggested that 'if two different physical systems are governed by the same differential equation, it's clear that there is some similarity between these systems, no matter how disparate the systems may seem … It seems plausible, at least, that this similarity is structural … ' (Colyvan 2001, p. 83).

Dimensional explanations, I have argued elsewhere (2009c), are another kind of non-causal scientific explanation that uses structural features of more fundamental equations to unify physically dissimilar phenomenological laws.

Some philosophers have argued that certain other physical facts are explained by mathematical theorems. (For discussion see: Baker 2005; Colyvan 2001 and 2002; Melia 2002; Steiner 1978b and references therein.) Steiner (1978b, p. 19) contends that if the physics is removed from such a mathematical explanation of a physical fact, then a mathematical explanation of some mathematical theorem remains, whereas '[i]n standard scientific explanations, after deleting the physics nothing remains'. I agree that if the physics is removed from an explanation of the similarity between these two phenomenological results, then the two results and their derivations become one. But if a mathematical theorem is explained by the argument thereby produced, then the same theorem must be explained by the argument that is produced when the physics is removed from a 'standard scientific explanation' of just one of these phenomenological results using the more fundamental equations that entail it (since the same mathematical derivation is produced in both instances). In that event, it would not be the case that nothing remains after deleting the physics from this standard scientific explanation.

Other scientific explanations work in the same way. For instance, consider the mathematical proof that Poisson (1811, pp. 11–19) uses to explain why two forces compose according to the 'parallelogram of forces'. Poisson's proof derives the parallelogram of forces from various symmetries that forces exhibit (such as that equal and opposite forces cancel). The same mathematical proof can also be used to deduce that various other directed quantities (such as electric current densities, bulk magnetizations, and entropy fluxes) also compose according to parallelogram laws since those quantities exhibit analogous symmetries. Thus, if all of these various parallelogram laws are explained by Poisson-style arguments, then it is no *mathematical* coincidence that these various quantities, despite their *physical* diversity, all compose in the same way:

> [T]he proof which Poisson gives of the 'parallelogram of forces' is applicable to the composition of any quantities such that turning them end for end is equivalent to a reversal of their sign. (Maxwell 1873, p. 10)

Even if there is no *common* reason why equal-and-opposites cancel for light, water, velocity, force, energy, electric current, and so forth, there is (if Poisson-style explanations are correct) a basic similarity among them that makes it no (mathematical) coincidence that they all compose parallelogramwise.[23]

Our account of mathematical coincidence has thus paid dividends in helping us to recognize one kind of understanding that certain scientific explanations supply. The concept of a mathematical coincidence plays an important role in scientific practice.

## 5. Conclusion

I think that G. H. Hardy is implicitly appealing to the notion of a mathematical coincidence when he praises theorems having proofs where

> There are no complications of detail – one line of attack is enough in each case … We do not want many 'variations' in the proof of a mathematical theorem: 'enumeration by cases', indeed, is one of the duller forms of mathematical argument (Hardy 1967, p. 113)

On my view, that is because a proof by cases (such as the first proof I gave of the summation formula) cannot show the theorem to be no

[23] For more on Poisson's argument (and further lessons to be learned from various rival explanations of the parallelogram of forces), see my 2009b.

coincidence. A proof by cases is the only explanation that a mathematical coincidence is capable of receiving.[24]

## References

Baker, Alan 2005: 'Are There Genuine Mathematical Explanations of Physical Phenomena?' *Mind*, 114, pp. 223–38.

Baker, Alan forthcoming: 'Mathematical Accidents and the End of Explanation'. Forthcoming in Bueno and Linnebo.

Bueno, Otávio and Øystein Linnebo (eds) forthcoming: *New Waves in the Philosophy of Mathematics*. Forthcoming from Palgrave Macmillan.

Colyvan, Mark 2001: *The Indispensability of Mathematics*. Oxford: Oxford University Press.

—— 2002: 'Mathematics and Aesthetic Considerations in Science'. *Mind*, 111, pp. 69–78.

Corfield, David 2005: 'Mathematical Kinds, or Being Kind to Mathematics'. *Philosophica*, 74, pp. 30–54.

Czermak, J. (ed.) 1993: *Philosophy of Mathematics, Proceedings of the 15th International Wittgenstein Symposium, Volume 1*. Vienna: Verlag Hölder-Pichler-Tempsky.

Davis, Philip J. 1981: 'Are There Coincidences in Mathematics?' *The American Mathematical Monthly*, 88, pp. 311–20.

Feynman, Richard, Robert Leighton, and Matthew Sands 1963: *The Feynman Lectures on Physics, Volume 2*. Reading MA: Addison-Wesley.

Gowers, William Timothy 2007: 'Mathematics, Memory, and Mental Arithmetic'. In Leng, Paseau, and Potter 2007, pp. 33–58.

Guy, Richard K. 1988: 'The Strong Law of Large Numbers'. *The American Mathematical Monthly*, 95, pp. 697–712.

Hafner, Johannes and Paolo Mancosu 2005: 'The Varieties of Mathematical Explanation'. In Mancosu, Jørgensen, and Pedersen 2005, pp. 215–50.

Hanna, Gila 1990: 'Some Pedagogical Aspects of Proof'. *Interchange*, 21, pp. 6–13.

[24] My thanks to audiences at North Carolina State University and the University of North Carolina at Chapel Hill, as well as to Tim Gowers, Chris Haufe, and Roy Sorensen. Thanks also to the referees for several valuable suggestions.

Hardy, G. H. 1967: *A Mathematician's Apology*. Cambridge: Cambridge University Press.
Hume, David 1980: *Dialogues Concerning Natural Religion*, ed. Richard Popkin. Indianapolis: Hackett.
Kitcher, Philip 1975: 'Bolzano's Ideal of Algebraic Analysis'. *Studies in History and Philosophy of Science*, 6, pp. 229–69.
—— 1976: 'Explanation, Conjunction and Unification'. *Journal of Philosophy*, 73, pp. 207–12.
—— 1982: 'Explanatory Unification'. *Philosophy of Science*, 48, pp. 507–31.
—— 1984: *The Nature of Mathematical Knowledge*. Oxford: Oxford University Press.
—— 1989: 'Explanatory Unification and the Causal Structure of the World'. In Kitcher and Salmon 1989, pp. 410–505.
Kitcher, Philip and Wesley Salmon (eds) 1989: *Scientific Explanation: Minnesota Studies in the Philosophy of Science, Volume XIII*. Minneapolis: University of Minnesota Press.
Lange, Marc 2000: *Natural Laws in Scientific Practice*. New York: Oxford University Press.
—— 2009a: 'Why Proofs by Mathematical Induction are Generally Not Explanatory'. *Analysis*, 69, pp. 203–11.
—— 2009b: 'A Tale of Two Vectors'. *Dialectica*, 63, pp. 397–431.
—— 2009c: 'Dimensional Explanations'. *Noûs*, 43, pp. 742–75.
Leavitt, David 2007: *The Indian Clerk*. New York: Bloomsbury USA.
Leng, Mary, Alexander Paseau, and Michael Potter (eds) 2007: *Mathematical Knowledge*. Oxford: Oxford University Press.
Lewis, David 1986a: *Philosophical Papers: Volume 2*. Cambridge: Cambridge University Press.
—— 1986b: 'Causal Explanation'. In Lewis 1986a, pp. 214–40.
Lord, Nick 2007: 'An Amusing Sequence of Trigonometric Integrals'. *The Mathematical Gazette*, 61(521), pp. 281–5.
Mancosu, Paolo (ed.) 2008a: *The Philosophy of Mathematical Practice*. Oxford: Oxford University Press.
—— 2008b: 'Mathematical Explanation: Why It Matters'. In Mancosu 2008a, pp. 134–50.
Mancosu, Paolo, Klaus Frovin Jørgensen, and Stig Andur Pedersen (eds) 2005: *Visualization, Explanation, and Reasoning Styles in Mathematics*. Dordrecht: Springer.
Maxwell, James Clerk 1873: *A Treatise on Electricity and Magnetism: Volume 1*. Oxford: Clarendon Press.

——1881: *An Elementary Treatise on Electricity*. Oxford: Clarendon Press.

Melia, Joseph 2002: 'Response to Colyvan'. *Mind*, 111, pp. 75–9.

Nummela, Eric 1987: 'No Coincidence'. *The Mathematical Gazette*, 71(456), 147.

Owens, David 1992: *Causes and Coincidences*. Cambridge: Cambridge University Press.

Poisson, Siméon Denis 1811: *Traité de Mécanique, Volume 1*. Paris: Courcier.

Potter, Michael 1993: 'Infinite Coincidences and Inaccessible Truths'. In Czermak 1993, pp. 307–13.

Resnik, Michael and David Kushner 1987: 'Explanation, Independence and Realism in Mathematics'. *British Journal for the Philosophy of Science*, 38, pp. 141–58.

Sloane, N. J. A. 2007: *The On-Line Encyclopedia of Integer Sequences*, < www.research.att.com/~njas/sequences/ >, accessed 24 August 2009.

Sorensen, Roy MS: 'Mathematical Coincidences'.

Spivak, Michael 1980: *Calculus*, 2nd edition. Berkeley: Publish or Perish.

Steiner, Mark 1978a: 'Mathematical Explanation'. *Philosophical Studies*, 34, pp. 135–51.

——1978b: 'Mathematics, Explanation, and Scientific Knowledge'. *Noûs*, 12, pp. 17–28.

——1990: 'Mathematical Autonomy'. *Iyyun*, 39, pp. 101–14.

Tappenden, Jamie 2005: 'Proof Style and Understanding in Mathematics I: Visualization, Unification, and Axiom Choice'. In Mancosu, Jørgensen, and Pedersen 2005, pp. 147–214.

——2008: 'Mathematical Concepts: Fruitfulness and Naturalness'. In Mancosu 2008a, pp. 276–301.

Thomson, William [Lord Kelvin] 1845: 'On the Mathematical Theory of Electricity in Equilibrium'. *Cambridge and Dublin Mathematical Journal*, 1, pp. 75–95, In Thomson 1872, pp. 15–37.

——1872: *Reprint of Papers on Electrostatics and Magnetism*. London: Macmillan.

van Spronsen, J. W. 1969: *The Periodic System of Chemical Elements*. Amsterdam: Elsevier.

Waismann, Friedrich 1982: *Lectures on the Philosophy of Mathematics*, ed. Wolfgang Grassl. Amsterdam: Rodopi.