Why proofs by mathematical induction are generally not explanatory

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Philosophers who regard some mathematical proofs as explaining why theorems hold, and others as merely proving that they do hold, disagree sharply about the explanatory value of proofs by mathematical induction. I offer an argument that aims to resolve this conflict of intuitions without making any controversial presuppositions about what mathematical explanations would be.

1. The problem

Proposed accounts of scientific explanation have long been tested against certain canonical examples. Various arguments are paradigmatically explanatory, such as Newton's explanations of Kepler's laws and the tides, Darwin's explanations of various biogeographical and anatomical facts, Wegener's explanation of the correspondence between the South American and African coastlines, and Einstein's explanation of the equality of inertial and gravitational mass. Any promising comprehensive theory of scientific explanation must deem these arguments to be genuinely explanatory. By the same token, there is widespread agreement that various other arguments are not explanatory – examples so standard that I need only give their familiar monikers: the flagpole, the eclipse, the barometer, the hexed salt and so forth. Although there are some controversial cases, of course (such as 'explanations' of the dormitive-virtue variety), philosophers who defend rival accounts of scientific explanation nevertheless agree to a large extent on the phenomena that they are trying to save.

Alas, the same cannot be said when it comes to mathematical explanation. Philosophers disagree sharply about which proofs of a given theorem explain *why* that theorem holds and which merely prove *that* it holds.[1] This kind of disagreement makes it difficult to test proposed accounts of mathematical explanation without making controversial presuppositions about the phenomena to be saved.

1 Resnik and Kushner (1987: 146), among others, 'have doubts that any proofs explain' insofar as 'explaining' involves something other than merely supplying information that is relevant to our interests in the given context, or perhaps doing so in a systematic way. However, I believe that detailed case studies, such as those by Hafner and Mancosu (2005), decisively refute Resnik's and Kushner's (1987: 151) claim that '[m]athematicians rarely describe themselves as explaining' and fail in practice to distinguish between explanatory and non-explanatory proofs.

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One especially striking disagreement concerns the explanatory power of proofs that proceed by mathematical induction.[2] Some philosophers appear quite confident that these arguments are generally explanatory, even in the face of other philosophers who appear equally confident of their intuition to the contrary. Very little in the way of argument is offered for either view. The face-off is philosophically sterile.

To provoke *your* intuitions, consider the following two typical proofs by mathematical induction:

Show that for any natural number n, the sum of the first n natural numbers is equal to $n(n+1)/2$.

For $n = 1$, the sum is 1, and $n(n+1)/2 = 1(2)/2 = 1$.

If the summation formula is correct for $n = k$, then the sum of the first $(k+1)$ natural numbers is $[k(k+1)/2] + (k+1) = (k+1)[(k/2)+1] = (k+1)(k+2)/2$, so the summation formula is correct for $n = k+1$.

Show that for any natural number n, the sum of the squares of the first n natural numbers is equal to $n^3/3 + n^2/2 + n/6$.

For $n = 1$, the sum is 1, and $n^3/3 + n^2/2 + n/6 = 1/3 + 1/2 + 1/6 = 1$.

If the summation formula is correct for $n = (k-1)$, then the sum of the squares of the first k natural numbers is $(k-1)^3/3 + (k-1)^2/2 + (k-1)/6 + k^2 = 1/3(k^3 - 3k^2 + 3k - 1) + 1/2(k^2 - 2k + 1) + 1/6(k - 1) + k^2$
$= k^3/3 - k^2 + k^2/2 + k^2 + k - k + k/6 - 1/3 + 1/2 - 1/6$
$= k^3/3 + k^2/2 + k/6$, so the summation rule is correct for $n = k$.

Proofs like these seem to provoke strong intuitions among some philosophers. For example, Kitcher writes:

> Suppose I prove a theorem by induction ... It would seem hard to deny that this is a genuine proof. ... Further, this type of proof does not controvert Bolzano's claim that genuine proofs are explanatory; ... the proof explains the theorem. (1975: 265)

Regarding the first proof by mathematical induction that I gave above, along with a proof by mathematical induction that the sum of the first n odd numbers (for any natural number n) is n^2, Brown writes

> In the two number theory cases above, a proof by induction is probably more insightful and explanatory than the picture proofs [that is, diagrams that prove the same theorems]. I suspect that induction – the passage from n to n + 1 – more than any other feature, best characterizes

2 Recall that arguments by mathematical induction are deductions – unlike, for example, the argument that since 1, 1 + 8, and 1 + 8 + 27 are all perfect squares, it is likely true that for any natural number n, the sum of the first n cubes is a perfect square. The latter argument is not a proof; it is not a case of 'mathematical induction' in the sense I mean here.

> the natural numbers. That's why a standard proof [that is, by mathematical induction rather than by diagrams] is in many ways better – it is more explanatory. (1997: 177; cf. Brown 1999: 42)

On the other hand, Steiner (1978: 151) says that proofs by mathematical induction 'usually' are not explanatory. Steiner is the only philosopher I know who has defended his view regarding the explanatory power (or impotence) of mathematical inductions by appealing to a comprehensive account of mathematical explanation. However, he offers no argument independent of that account for his view concerning mathematical inductions. So it is difficult to regard him as testing his account by seeing whether it gives the right answer regarding mathematical inductions.

Writing about the first inductive proof above, Hafner and Mancosu (2005: 234) say that to deem it explanatory 'would indeed be very counterintuitive!' They elaborate slightly:

> it clearly isn't [explanatory] ... according to the understanding of working mathematicians (some mathematicians even take inductive proofs to be *paradigms of non-explanatory* proofs). (Hafner and Mancosu 2005: 237)

But they offer neither references to mathematicians expressing this view nor an argument to motivate it. Likewise, regarding the first induction I gave above, Hanna (1990: 10; cf. Hanna 1989: 48) writes, '[T]his is certainly an acceptable proof ... What it does not do, however, is show *why* the sum of the first n integers is $n(n+1)/2$ ... Proofs by mathematical induction are non-explanatory in general.'[3] There is also a small body of empirical psychological studies (e.g. Reid 2001; Smith 2006) suggesting that students generally regard proofs by mathematical induction as deficient in *explaining why* the theorem proved is true.

My aim in this brief article is to end this fruitless exchange of intuitions with a neat argument that proofs by mathematical induction are generally not explanatory. Although this argument is very simple, it does not appear in the literature. To recognize why mathematical inductions are generally not explanatory, we do not need to join Brown in considering what feature best characterizes the natural numbers or to join Steiner in appealing to some controversial premisses about how mathematical explanations operate – just as we do not need to appeal to controversial premisses about scientific explanation in order to recognize certain canonical examples as genuine scientific explanations (and others as possessing no explanatory power).

3 Mancosu (2001: 113) notes the contrast between Hanna and Brown. Of course, it could be that all of these writers are mistaken because some mathematical inductions are explanatory, others are not, and there is no broad truth about what they 'usually' or 'generally' are.

Here is all that I shall presuppose about mathematical explanation. First, I shall presuppose that a mathematical explanation of a given mathematical truth F may consist of a proof (i.e. a deduction) of F from various other mathematical truths $G_1, \ldots, G_n$. (If no mathematical proof could be a mathematical explanation, then proofs by mathematical induction would be non-starters as mathematical explanations.) In such an explanation, the G_i collectively explain why F obtains; each of G_1, G_2 and so forth helps to explain F (e.g. F is explained partly by G_1). Such a mathematical explanation is like many typical scientific explanations. For example, Kepler's laws of planetary motion are explained by being deduced from Newton's laws of motion and gravity, that the sun and planets are roughly spherical hunks of matter, that the sun contains nearly all of the solar system's matter, and so forth; hence, Newton's law of gravity helps to explain why Kepler's laws hold.[4] Accordingly, I will take over the terminology standardly used in discussing scientific explanations and say that in such a mathematical explanation, the G_i (the explainers) constitute the 'explanans' and F (the fact being explained) is the 'explanandum'.

Philosophers who disagree sharply about which mathematical proofs are explanatory nevertheless agree that not all mathematical proofs are mathematical explanations; indeed, they say, of two proofs having exactly the same premisses and conclusion, one may be explanatory even though the other is not. Regarding the crucial question of what it takes for various mathematical truths G_i to constitute the explanans in such a mathematical explanation of F, I shall presuppose very little about the answer (beyond F's following deductively from the G_i). In fact, I shall presuppose only that mathematical explanations cannot run in a circle. That is, I will presuppose that when one mathematical truth helps to explain another, the former is partly *responsible* for the latter in such a way that the latter cannot then be partly responsible for the former. Relations of explanatory priority are asymmetric. Otherwise mathematical explanation would be nothing at all like scientific explanation.

Of course, this presupposition permits a given mathematical truth to have many different explanations. It also permits mathematical *definitions* to run in circles, since a definition of some mathematical truth F need not be specifying *why* F is true. (Admittedly, in asking for a definition of F, we might say, 'Could you please explain F?' But that question need not be demanding a mathematical explanation, i.e. an answer to the question 'Why is F true?') If mathematical 'explanations' do not trace asymmetric relations of explanatory priority, then they merely display illuminating connections among various mathematical facts. They do not answer why-questions. Although it is

4 Of course, not every scientific explanation is a deductive argument for the fact being explained; in some cases, the explainers merely entail that the fact being explained has a given likelihood. Moreover, neither Newton's laws nor Kepler's laws are exactly true. But we do not need to worry about either of these complications in the mathematical case.

valuable to get clear about how the truth of various propositions in one part of the mathematical landscape connects to the truth of propositions in another, those connections need not be mathematical *explanations* of one by the other.

Plausibly, that F explains why G holds and that G explains why H holds entails that F explains why H holds. In that case, to permit an explanatory circle (where F explains why G holds and G explains why F holds) would be to permit a given fact to explain itself. But self-explanation in mathematics is presumably impossible. Here, then, is further support for my presumption that mathematical explanations, whatever they are, cannot run in a circle.

2. *The proposed solution*

A proof by mathematical induction proceeds according to the following rule of inference:

Mathematical induction:

For any property P:
if P(1) [1 has property P] and
for any natural number (i.e. 1, 2, 3...) k, if P(k), then P($k+1$).
then for any natural number n, P(n).

Since this rule of inference is necessarily truth-preserving, an argument by mathematical induction constitutes a proof.

Is such an argument explanatory? If it were, what would the mathematical explanation be? The explanans would be (for some particular property P) the fact that P(1) and that (for any natural number k) if P(k) then P($k+1$). For instance, if the first proof given above were explanatory, then the explanandum – the fact that for any natural number n, the sum of the first n natural numbers is $n(n+1)/2$ – would be explained in part by the fact that the summation formula works for $n=1$.

But the case of $n=1$, though often more mathematically tractable than other cases, seems to have no special explanatory privilege over them. From this thought, I will now work my way towards an argument that mathematical inductions are generally not explanatory.

If a theorem can be proved by mathematical induction proceeding upwards from $n=1$, then generally it can also be proved by an argument like mathematical induction, but instead proceeding upwards and downwards from $n=k$ for any other, arbitrarily selected natural number k. Of course, it is usually easier to begin by showing that P(1) rather than by showing, say, that P(5), since the proof of P(1) is often trivial. And it is obviously easier to show merely that if P(k), then P($k+1$), rather than to demonstrate not only this 'upwards' fact, but also the 'downwards' fact that if P(k), then P($k-1$), for any natural number $k>1$. (A proof cannot be given by proceeding exclusively downwards, since there is no largest natural number from which to proceed.) But the longer argument would be just as effective as the argument by mathematical induction in proving that P(n) for any natural number n. The rule of inference:

Upwards and downwards from 5:

For any property P:
if P(5), and
for any natural number k, if P(k), then P($k+1$), and
for any natural number $k>1$, if P(k), then P($k-1$),
then for any natural number n, P(n)

is necessarily truth-preserving, just like the rule of mathematical induction.[5] Indeed, the term 'mathematical induction' is sometimes applied to arguments that use not the rule of mathematical induction given above, but some other necessarily truth-preserving rule belonging to the same family as that rule and the 'upwards and downwards from 5' rule, such as:

Upward jumps of 2:

For any property P:
if P(1) and P(2), and
for any natural number k, if P(k), then P($k+2$),
then for any natural number n, P(n).

Strong induction:

For any property P:
if P(1), and
for any natural number k, if P(1) and P(2) and ... P($k-1$), then P(k),
then for any natural number n, P(n).

Backward induction (Hardy et al. 1934: 20):

For any property P:
if P(k) holds for an infinite number of natural numbers k, and
for any natural number k, if P(k), then P($k-1$).
then for any natural number n, P(n).

Let us stick to the 'upwards and downwards from 5' rule. If a theorem can be proved by mathematical induction (i.e. by the 'upwards from 1' rule), then generally it can also be proved by an argument using the 'upwards and downwards from 5' rule. For instance, the 'upwards and downwards from 5' rule can be used to prove the two theorems that were proved above by mathematical induction:

Show that for any natural number n, the sum of the first n natural numbers is equal to $n(n+1)/2$.

5 The 'upwards and downwards from 5' rule does not have us check by hand that P(1) through P(5) are all true. Rather, it proceeds from P(5) downward as well as upward by an 'inductive-style' argument.

For $n=5$, the sum is $1+2+3+4+5=15$, and $n(n+1)/2=5(6)/2=15$.

If the summation formula is correct for $n=k$, then (I showed earlier) it is correct for $n=k+1$.

If the summation formula is correct for $n=k$ (where $k>1$), then the sum of the first $k-1$ natural numbers is $[k(k+1)/2]-k=k[(k+1)/2-1]=k(k-1)/2$, so the summation formula is correct for $n=k-1$.

Show that for any natural number n, the sum of the squares of the first n natural numbers is equal to $n^3/3+n^2/2+n/6$.

For $n=5$, the sum is $1+4+9+16+25=55$, and $n^3/3+n^2/2+n/6=125/3+25/2+5/6=41\ 2/3+12\ 1/2+5/6=55$.

If the summation formula is correct for $n=k$, then (I showed earlier) it is correct for $n=k+1$.

If the summation formula is correct for $n=k$ (where $k>1$), then the sum of the squares of the first k–1 natural numbers is

$$\begin{aligned}
&k^3/3+k^2/2+k/6-k^2\\
&=k^3/3+k^2/2+k/6-k^2+k-k-1/3+1/2-1/6\\
&=k^3/3-k^2+k-1/3+k^2/2-k+1/2+k/6-1/6\\
&=1/3\ (k^3-3k^2+3k-1)+1/2\ (k^2-2k+1)+1/6\ (k-1)\\
&=(k-1)^3/3+(k-1)^2/2+(k-1)/6,
\end{aligned}$$

so the summation rule is correct for $n=(k-1)$.

If the proofs by mathematical induction are explanatory, then the very similar proofs by the 'upwards and downwards from 5' rule are equally explanatory. There is nothing to distinguish them, except for where they start. But they cannot *both* be explanatory. It cannot be that P(1) helps to explain why P(5) holds *and* that P(5) helps to explain why P(1) holds, on pain of mathematical explanations running in a circle. Therefore, a mathematical induction does not explain the theorem that it proves if that theorem can also be proved by the 'upwards and downwards from 5' rule. Since generally any theorem provable by mathematical induction can also be proved by the 'upwards and downwards from 5' rule, mathematical inductions are generally not explanatory.

This argument does not show merely that *some* proofs by mathematical induction are not explanatory. It shows that *none* are – because if one were explanatory, then the corresponding proof by the 'upwards and downwards from 5' rule would also be explanatory, and they cannot both be. It would be arbitrary for one of these arguments but not the other to be explanatory.

It might be objected that there is no circularity involved in both of these arguments being explanatory: the argument by mathematical induction explains why it is the case that for any n, P(n); it does not explain why P(5). So when the argument by the 'upwards and downwards from 5' rule uses P(5) to explain the fact that for any n, P(n), the explanans is not a fact that the inductive argument explains.

I reply: if the argument from mathematical induction uses P(1) to explain why it is the case that for any n, P(n), then for any $n \neq 1$, the argument uses P(1) to explain why P(n) is true – and so, in particular, uses P(1) to explain why P(5) is true. Compare a scientific example: Coulomb's law (giving the electrostatic force between two stationary point charges) explains why the magnitude E of the electric field of a long straight wire (of negligible thickness) with uniform linear charge density λ is $2\lambda/r$ at a distance r from the wire. In explaining why for any λ and r, $E = 2\lambda/r$, Coulomb's law explains in particular why $E = 4$ dyn/statcoulomb if $\lambda = 10$ statcoulombs/cm and $r = 5$ cm. By the same token, if P(1) explains why for any n, P(n), then P(1) explains in particular why P(5).

It might be suggested that although my argument shows *that* mathematical induction is generally not explanatory (on pain of explanatory circularity), my argument fails to show *why* it is generally not explanatory. But it does suggest why. An explanation by mathematical induction would have to include P(1) as part of the explanans. However, P(1) is generally not explanatorily prior to P(5), for example, or to any other instance of the theorem being proved. It would be arbitrary to privilege P(1). So mathematical inductions are generally not explanatory.[6]

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6 Some philosophers may believe that 1 is ontologically prior to the other natural numbers – for instance, that every natural number is somehow constituted by 1. For all I have said, such philosophers may take mathematical inductions to be explanatory. Whatever argument these philosophers give that 1 is prior would break the symmetry between a proof by mathematical induction and a proof of the same theorem that proceeds upwards and downwards from 5. I have shown only that a commitment to the explanatory power of mathematical inductions would require a commitment to 1's ontological priority over the other natural numbers. Although such priority seems dubious to me, I have offered no argument against it.

Of course, philosophers dubious of mathematical explanation in general might try to generalize my circularity worry: taking any proof to be explanatory requires arbitrarily privileging some particular axiomatization of mathematics.

Suppose we wanted to know not 'Why is it the case that for any natural number n, P(n)?', but rather 'Why does P(n) hold either for any natural number n or for no natural number at all?' (In other words, 'Why is it *not* the case that P(n) holds for some but not for all natural numbers n?') Perhaps an argument in the same family as mathematical induction, showing that we can prove this result by proceeding upwards and downwards from any arbitrary value for n, could answer *this* question. Since this argument allows any initial value for n, it does not arbitrarily privilege the fact that P(k) for some particular k. To answer the question, the argument would have to show only that *if* P(n) holds for one natural number n, then P(n) holds for all natural numbers. So only the 'inductive step' of the induction-style argument would be needed.

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Classical logic without bivalence

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1. Introduction

Semantic justifications of the classical rules of logical inference typically make use of a notion of bivalent truth, understood as a property guaranteed to attach to a sentence or its negation regardless of the prospects for speakers to determine it as so doing. For want of a convincing alternative account of classical logic, some philosophers suspicious of such recognition-transcending bivalence have seen no choice but to declare classical deduction unwarranted and settle for a weaker system; intuitionistic logic in particular, buttressed by assertion-conditional semantics, is often considered to enjoy a degree of meaning-theoretical respectability unattainable by classical logic.

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