Erkenn (2018) 83:1285–1302
https://doi.org/10.1007/s10670-017-9941-z

ORIGINAL RESEARCH

Mathematical Explanations that are Not Proofs

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Received: 26 January 2017 / Accepted: 15 September 2017 / Published online: 25 September 2017

Abstract Explanation in mathematics has recently attracted increased attention from philosophers. The central issue is taken to be how to distinguish between two types of mathematical proofs: those that explain why what they prove is true and those that merely prove theorems without explaining why they are true. This way of framing the issue neglects the possibility of mathematical explanations that are not proofs at all. This paper addresses what it would take for a non-proof to explain. The paper focuses on a particular example of an explanatory non-proof: an argument that mathematicians regard as explaining why a given theorem holds regarding the derivative of an infinite sum of differentiable functions. The paper contrasts this explanatory non-proof with various non-explanatory proofs (and non-explanatory nonproofs) of the same theorem. The paper offers an account of what makes the given non-proof explanatory. This account is motivated by investigating the difficulties that arise when we try to extend Mark Steiner's influential account of explanatory proofs to cover this explanatory non-proof.

1 Introduction

Compared to scientific explanation, which has been a central topic in the philosophy of science for over sixty years, explanation in mathematics has been relatively understudied. However, philosophers are now paying increased attention to the ways in which mathematicians explain why a given mathematical fact holds (Leng 2005; Mancosu 2008; Baker 2009). These philosophers have invariably focused

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their attention on the contrast between two different classes of mathematical proofs: those that explain what they prove and those that do not. Often the contrast is drawn between proofs of the very same theorem—one explaining why it holds and another merely proving that it holds. The philosophical goal is to identify the source of this difference in the proofs' explanatory power (just as accounts of scientific explanation aim to understand why, for instance, an inference from the flagpole's height can explain its shadow's length, but an inference from the shadow to the flagpole is not explanatory). Researchers in mathematics education (e.g., Mudaly and de Villiers 2000; Healy and Hoyles 2000) have also investigated empirically how learners assess the explanatory power of different proofs of the same theorem and how students seek a proof that explains why a result is true even if they have already seen a proof that fully convinces them that it is true.

Here is a lovely example from the mathematical literature (first brought to my attention by Roy Sorensen).[1] Take an ordinary calculator keyboard.

7	8	9
4	5	6
1	2	3

We can form a six-digit number by taking the three digits on any row, column, or main diagonal on the keyboard in forward and then in reverse order. For instance, the bottom row taken from left to right, and then right to left, yields 123321. There are sixteen such "calculator numbers": 123321, 321123, 456654, 654456, 789987, 987789, 147741, 741147, 258852, 852258, 369963, 963369, 159951, 951159, 357753, and 753357. As you can easily verify (with a calculator!), every one of these numbers is divisible by 37. Is this (as the title of a recent *Mathematical Gazette* article asks) a coincidence?[2] Why does the calculator-number theorem hold?

A proof that simply takes each calculator number in turn, showing it to be divisible by 37, treats the result as if it were a coincidence. In contrast, here is a proof of the calculator-number theorem (from a later *Mathematical Gazette* article entitled "No Coincidence") that explains why it holds:

Let a, a + d, a + 2d be any three integers in arithmetic progression. Then

$$\begin{aligned} & a \cdot 10^5 + (a + d) \cdot 10^4 + (a + 2d) \cdot 10^3 + (a + 2d) \cdot 10^2 + (a + d) \cdot 10 + a \cdot 1 \\ &= a\left(10^5 + 10^4 + 10^3 + 10^2 + 10 + 1\right) + d\left(10^4 + 2 \cdot 10^3 + 2 \cdot 10^2 + 10\right) \\ &= 111111a + 12210d = 1221(91a + 10d). \end{aligned}$$

[1] I discuss this example and Sorensen's views in my (2010) and (2016).

[2] The article appears (unsigned, as a "gleaning") on p. 283 of the December 1986 issue.

> So not only is the number divisible by 37, but by 1221 (= 3 × 11 × 37) (Nummela 1987: 147)

An account of mathematical explanation should identify what makes the latter proof explanatory, unlike the brute-force proof that checks each calculator number individually.

However, not all mathematical explanations consist of proofs of the facts they explain.[3] Although mathematical explanations that are not proofs have received very little of the attention that philosophers (and empirical researchers in mathematics education) have recently paid to mathematical explanation, any adequate theory of mathematical explanation must account for mathematical explanations that are not proofs (just as an adequate theory of scientific explanation must account for those scientific explanations, such as statistical explanations, where the explanans fails to logically entail the explanandum).[4]

In this paper, I will focus on one example of this neglected variety of mathematical explanation. The fact being explained can be proved, but the typical textbook proof does not explain why it holds, whereas an argument that is not a proof succeeds in explaining it. Instead of contrasting an explanatory proof with a non-explanatory proof and trying to account for which is which, I will contrast a non-explanatory proof with an explanatory non-proof (as well as with several non-explanatory non-proofs) and I will try to account for their differences in explanatory power.[5] In Sect. 2, I will give the example. In Sect. 3, I will explore whether one influential account of explanatory proofs (Mark Steiner's) can be naturally expanded to cover it, drawing some lessons from that effort. Then, in Sect. 4, I will propose my own account.

Though I will focus primarily on this single example, I believe that many other mathematical explanations work in the same way as it does. In Sect. 5, I will briefly mention some other examples. They provide new and important ways of testing any proposed account of mathematical explanation. How such non-proofs explain can thus shed light on why some proofs rather than others explain.

[3] Of course, there are many senses of "mathematical explanation" that do not involve proofs; for instance, I can explain to my class what "uniform convergence" means or how to evaluate an integral or why so many students got the wrong answer on their test. None of these is a "mathematical explanation" in the sense of this paper (and of the recent literature I just mentioned). The same point arises in connection with scientific explanation; as Hempel (2001: 80) pointed out, an account of scientific explanation does not aim to cover "the vastly different senses of 'explain' involved when we speak of explaining the rules of a game, or … when we ask someone to explain to us how to repair a leaking faucet".

[4] Of course, a mathematical explanation may fail to prove the fact it explains because it merely sketches such a proof rather than giving it fully—just as some "scientific explanations" are merely explanation sketches. My concern in this paper is with some mathematical explanations that do not work by proving or even by sketching a proof of their explanandum.

[5] D'Alessandro (forthcoming) also discusses mathematical non-proofs that explain. He focuses on *non-arguments* that explain in mathematics; in particular, he argues that various mathematical theorems explain, independent of their proofs. By contrast, I will focus on mathematical *arguments* that are not proofs but nevertheless explain.

2 A Mathematical Explanation that Does Not Prove Its Explanandum

Take an infinite sequence of functions $f_1(x)$, $f_2(x)$, $f_3(x)$…, where each function is differentiable at $x = a$. Consider the function $(f_1 + f_2 + f_3 + \cdots)(x)$, i.e., the function that is the sum $f_1(x) + f_2(x) + f_3(x)\cdots$. It turns out that for some $x = a$, the sum of the derivatives of the various terms does not equal the derivative of the sum. In other words, sometimes $f_1{}'(a) + f_2{}'(a) + f_3{}'(a)\cdots$ does not equal $(f_1 + f_2 + f_3 + \cdots)'(a)$. This theorem (that in some cases, the derivative of the infinite sum is unequal to the infinite sum of the derivatives) is our explanandum.

Obviously, a single example where these two quantities are unequal suffices to prove that they are not always equal. Here is such a typical textbook "proof by example". Consider the Fourier series

$$F(x) = \frac{4}{\pi}\left[\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) - \frac{1}{7}\cos\left(\frac{7\pi x}{2}\right) + \cdots\right]$$

For any $-1 < x < 1$, $F(x) = 1$, and so its derivative $F'(x) = 0$. However, differentiating F(x) term by term, we obtain

$$G(x) = -2[\sin(\pi x/2) - \sin(3\pi x/2) + \sin(5\pi x/2) - \sin(7\pi x/2) + \cdots]$$

and so, for instance,

$$G(.5) = -\sqrt{2}[1 - 1 - 1 + 1 + 1 - 1 - 1 + 1 + \cdots],$$

which fails to converge. Therefore, $G(.5) \neq F'(.5)$; we have proved by example that the derivative at $x = a$ of a sum of infinitely many terms, each differentiable at $x = a$, need not equal the sum of the terms' derivatives at $x = a$.[6]

Why is that? Why are these two quantities not always equal?

I am not sure that this is a well-posed question simply as it stands. As yet, there seems to me no clear sense to asking for an explanation of this theorem's truth if we do not intend to be asking merely for a proof of its truth. To ask why $G(.5) \neq F'(.5)$ in the above example seems to me nothing more than a potentially misleading way of asking for a proof that they are unequal. It would be like asking why $\int_1^3 (x^3 - 5x + 2)dx = 4$. When this equation is stated outside of any context, I can't see anything that it would be to explain why it holds; all we can do is to prove that it does.

However, we can take our theorem [that $(f_1 + f_2 + f_3 + \cdots)'(a)$ is not always equal to $f_1{}'(a) + f_2{}'(a) + f_3{}'(a)\cdots$] and place it in a richer context. If *two* functions f(x) and g(x) are each differentiable at $x = a$, then the function that is their sum $(f + g)(x) = f(x) + g(x)$ is also differentiable at $x = a$, and the derivative of the sum is the sum of the derivatives: $(f + g)'(a) = f'(a) + g'(a)$. The analogous theorem applies to the sum of *three* differentiable functions, or *four* differentiable

[6] The mathematical discussion here and below closely follows Bressoud (1994: 73–74). Note that if each of the f_i is differentiable on [a, b] and $f_1{}'(x) + f_2{}'(x) + f_3{}'(x)\cdots$ converges uniformly on [a, b], then $f_1{}'(x) + f_2{}'(x) + f_3{}'(x)\cdots = (f_1 + f_2 + f_3 + \cdots)'(x)$ for all x in [a, b].

functions—or *n* differentiable functions for any natural number *n*. This is the "sum rule" for differentiation. However, as we just saw, the analogous theorem does not apply in the infinite case. Why is that? Why doesn't the sum rule hold in the infinite case?

We now have a well-posed why question regarding our theorem. The contrast between the finite and infinite cases succeeds in fixing what it would be to explain why our theorem regarding the infinite case holds. Roughly speaking, an explanation of the theorem's holding must explain why the infinite case differs from the finite case with regard to the derivative of the sum equaling the sum of the derivatives. In other words, an explanation must (roughly speaking) reveal the difference it makes that the sum is infinite rather than finite. This is how the explanatory non-proof that I am about to give is often introduced in mathematics textbooks. For instance, Bressoud (1994: 65) presents the non-proof that I am about to give as answering the question "What goes wrong with the infinite series?", where "going wrong" is differing from finite series in sometimes violating the sum rule. I take Bressoud's question to be a demand for a mathematical explanation (as we will see shortly, he says explicitly that we can use the non-proof to "explain why" our explanandum holds). It is standard for the sum rule's failure in the infinite case to be presented as standing in stark contrast to the finite cases (see for instance, D'Angelo 2002: 23; Smith and McLelland 2003: 93). That our explanandum's salient feature is its contrast with the sum rule in finite cases will eventually play an important part in my account.

The "proof by example" given above does not explain why the sum rule fails in the infinite case; it merely shows that sometimes in the infinite case, the derivative of the sum does not equal the sum of the derivatives. Here, by contrast, is an explanation:

> Let's show that if f and g are differentiable at $x = a$, then $(f + g)'(a) = f'(a) + g'(a)$. By definition, $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ and $g'(a) = \lim_{h \to 0} \frac{g(a+h)-g(a)}{h}$. Let's unpack these limits: for every $\varepsilon/2 > 0$, there are δ_1, $\delta_2 > 0$ such that if $|h| < \delta_1$, then $|f'(a) - \frac{f(a+h)-f(a)}{h}| < \varepsilon/2$, and if $|h| < \delta_2$, then $|g'(a) - \frac{g(a+h)-g(a)}{h}| < \varepsilon/2$. Let δ be the smaller of δ_1 and δ_2 (or their common value if they are equal). Then for any $\varepsilon/2 > 0$, if $|h| < \delta$, then $|f'(a) + g'(a) - \frac{(f+g)(a+h)-(f+g)(a)}{h}| = |f'(a) - \frac{f(a+h)-f(a)}{h} + g'(a) - \frac{g(a+h)-g(a)}{h}| \leq |f'(a) - \frac{f(a+h)-f(a)}{h}| + |g'(a) - \frac{g(a+h)-g(a)}{h}| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. So $f'(a) + g'(a) = \lim_{h \to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h}$, which (by definition) equals $(f + g)'(a)$.
>
> Thus we have proved that for any two differentiable functions, the derivative of their sum is equal to the sum of their derivatives. To prove the analogous result for *n* functions, we replace $\varepsilon/2$ with ε/n and let δ be the member of $\delta_1,\ldots,\delta_n$ that is smaller than (or as small as) the rest; from among any finite number *n* of positive numbers, there is certain to be one that is smaller than (or as small as) each of the others. But not so for an infinite number of positive numbers! With infinitely many functions and hence infinitely many δ_i, there is

no guarantee that some positive number is less than or equal to every δ_i. Rather, the δ_i may approach arbitrarily close to 0.

That difference explains why in the infinite case, $(f_1 + f_2 + f_3 + \cdots)'(a)$ is not always equal to $f_1'(a) + f_2'(a) + f_3'(a)\cdots$. That this argument accounts for our theorem is not merely my own view. Mathematicians also characterize it as explanatory. For example, Bressoud (1994: 69) gives the above argument and says that we can thereby "explain why it is that sometimes you can differentiate an infinite series by differentiating each term, and sometimes you cannot".

Though explaining why the sum rule fails in the infinite case, the above argument does not prove this theorem. It reveals how a certain proof of the sum rule in every finite case cannot be generalized to the infinite case; it identifies a place in that proof where the existence of finitely many terms in the sum is needed. In particular, with infinitely many δ_i, it does not follow automatically that some positive number is less than or equal to every δ_i. However, the above argument does not demonstrate that there exists an infinite sum of differentiable functions where the δ_i approach arbitrarily close to 0 (of course, the "proof by example" reveals that there is). That the equation's proof in the finite case does not carry over to the infinite case fails to show that there is no proof of some other kind for the sum rule in the infinite case; it does not show that the equality fails in every infinite case or even that it fails in some infinite case. It shows merely that the door is open to its failing—not that any case walks through the open door. So here we have an explanation of our theorem that does not consist of a proof of the explanandum (though it does include a proof of the corresponding theorem for finite *n*).

3 Lessons from the Failure of a Steineresque Account

Mathematical explanations that are not proofs pose a new and important challenge to existing philosophical accounts of mathematical explanation. Perhaps the best-known account is Steiner's (1978). Like every other proposal in the literature, Steiner's account aims to distinguish "proofs that explain" from "proofs that merely demonstrate" (Steiner 1978: 135). In this section, I will examine whether Steiner's account can be expanded to cover the explanatory non-proof given in the previous section. The prospects initially seem favorable. Steiner's account (roughly speaking) identifies proofs as having explanatory power if and only if they generalize, and as we saw, the explanatory non-proof in the previous section also has something to do with generalizability: it works precisely by revealing a certain proof's generalizability to be limited. Therefore, Steiner's approach appears well-suited to capturing the source of this non-proof's explanatory power. However, I will uncover two obstacles to expanding Steiner's account to cover this explanatory non-proof. These obstacles will reveal some additional aspects of this explanation that make crucial contributions to its explanatory power. They will point us toward the proposal that I will make in the next section.

According to Steiner (1978), a proof of some theorem concerning S_1's (some class of mathematical entities) explains why that theorem holds if and only if the proof reveals how the theorem depends on S_1's "characterizing property"—that is, on the property essential to being an S_1 that is just sufficient to distinguish S_1's from other entities in the same "family" (for example, to distinguish triangles from other kinds of polygons). To reveal the theorem's dependence on S_1's characterizing property, the proof must be "generalizable" in the following sense. Suppose that S_1's characterizing property is replaced in the proof by the characterizing property for another kind S_2 in the same family, but otherwise the original "proof idea" is retained. Then the resulting "deformation" of the original proof proves another theorem—one concerning S_2's. In short, a proof of the theorem explains why the theorem holds exactly when it employs a strategy by which different, but analogous theorems can be proved for different classes in the same family.

As it stands, this account fails to cover the explanation given in the previous section. That explanation is not a proof, so perforce it is not a proof that generalizes. Some philosophers (e.g., Resnik and Kushner 1987: 147–152) have criticized Steiner's account on the grounds that certain explanatory proofs simply collapse when one tries to generalize them to other classes in the same family. The explanation from the previous section prompts a different objection to Steiner's account: if a proof of some theorem collapses when we try to generalize that proof to another class in the same family, then that proof's collapse can itself explain a mathematical fact about that other class. In other words, whereas Steiner regards a proof's *generalizability* as giving it the power to explain the theorem proved, it turns out that a proof's *failure to generalize* from S_1's to S_2's can explain why there is no theorem for S_2's analogous to the theorem proved for S_1's.

We could conclude at this point that Steiner's proposal simply cannot account for the explanation in the previous section. But let's not be hasty. Steiner acknowledges (Steiner 1978: 147) that not all mathematical explanations are proofs and suggests briefly that explanatory non-proofs are similar in important ways to explanatory proofs. Steiner does not work to elaborate his account of explanatory proofs so as to cover explanatory non-proofs. Let's see what happens if we try to do so.

Steiner's basic proposal is that a theorem's explanation reveals its dependence on the characterizing property of the class it concerns. Steiner identifies a proof's "generalizability" as the way in which it reveals such dependence. But perhaps dependence can also be revealed by a proof's non-generalizability. That the proofs of the sum rule for two functions, for three functions, and so forth fail to generalize to the sum of infinitely many functions explains why the sum rule fails to hold in the infinite case.

Of course, we had better not expand Steiner's proposal to say that a given theorem's proof explains why that theorem holds if and only if the proof either can be generalized or cannot be generalized. However, the explanation in the previous section did not consist of a proof *of the theorem being explained* that failed to generalize. Rather, it consisted of several proofs all using the same "proof idea" (as Steiner's would say), each proving a different theorem ("The derivative of the sum of *two* functions equals …", "The derivative of the sum of *three* functions equals …") that is *not* the theorem being explained ("It is not always the case that the derivative of the sum of infinitely many functions equals …"). All of these theorems (those proved as well as the explanandum) concern different classes in the same family. These proofs fail to generalize to prove the negation of the theorem being explained (i.e., to prove that the sum rule holds in the infinite case). Suppose we try expanding Steiner's account so that it deems to be explanatory any non-proof satisfying the conditions in the preceding three sentences. This expansion apparently expresses the notion that non-generalizability can sometimes explain.

To explain a given theorem, according to this "Steineresque" proposal, it does not suffice merely to offer a proof of another theorem (or a collection of proofs of several other theorems, all using the same "proof idea") where that proof fails to generalize to prove the explanandum's negation. The explanandum's negation is false, so obviously *every* proof fails to generalize to prove it. Yet only the failure of *certain* proofs to generalize to prove the explanandum's negation sheds any light on why the explanandum holds. For instance, consider the proof (in Sect. 1) that explains why the "calculator number" theorem holds. That proof (like every other proof of anything) fails to generalize to prove that the sum rule holds in the infinite case. But the Steineresque proposal does not say that the proof explaining the calculator-number theorem also explains why the sum rule fails in the infinite case. To do that, the calculator-number theorem would have to concern a class in the same family as the class of sums of infinitely many differentiable functions. Although Steiner (1978: 147) admits to having no definition of what it is for two classes to belong to the same "family", we can grant that the class of calculator numbers does not belong to the same family as the class of sums of infinitely many differentiable functions. In contrast, the class of sums of infinitely many differentiable functions arguably belongs to the same family as the class of sums of two differentiable functions, the class of sums of three differentiable functions, and so forth.

We need to refine this Steineresque proposal further (though still in the spirit of Steiner's original account) to give it the best opportunity of capturing the power of the non-proof (involving infinitely many δ_i) to explain why the sum rule fails in the infinite case. So far, the Steineresque proposal is satisfied by proofs (sharing a "proof idea") of any theorems at all regarding the sums of finitely many differentiable functions; those sums are arguably members of the same "family" as sums of infinitely many differentiable functions. But to explain why the sum rule fails in the infinite case, those proofs must succeed in proving not just any theorems at all regarding the sums of finitely many differentiable functions, but in particular theorems that are *analogous* to the fact being explained—in the way that various theorems are analogous if they specify that the sum rule holds (or fails to hold) for some range of cases. That the sum rule holds for any finite number of differentiable functions—and can be proved to do so by using a certain "proof idea"—might reasonably have suggested (at least to some degree) that the sum rule's holding in the infinite case can be proved in the same way. That (surprisingly) it cannot be so proved may thus help to explain why the sum rule fails in the infinite case.[7]

As I mentioned, Steiner's account of explanatory proofs says that a given proof explains when the same "proof idea" can be used to prove other theorems regarding

[7] *May* explain—but may not. As we will see, the explanation requires certain further conditions to be satisfied.

different classes in the same family. Implicit in the same proof idea's being used is arguably that these other theorems are analogous to the theorem being explained—that the theorems are all "deformations" of one another. So far, then, this proposal remains in the spirit of Steiner's own.

However, this Steineresque proposal fails to capture one important aspect of the relation that we have noticed between the explanandum (the sum rule's failure in the infinite case) and the sum rule's holding in all finite cases. Consider the argument involving infinitely many δ_i that explains why (without proving that) the sum rule fails in the infinite case. This argument's explanatory power depends (as I have said) upon the surprisingness of the sum rule's failure in the infinite case, considering that it holds in all finite cases. We saw earlier that even to ask for an *explanation* of the sum rule's failure in the infinite case, as distinct from a *proof* of its failure, makes sense only in a context where the sum rule's failure in the infinite case is presented as contrasting sharply with the sum rule's holding in every finite case. By contrast, Steiner maintains that a why question regarding our explanandum is meaningful independent of the explanandum's being juxtaposed to the sum rule's holding in every finite case. I have suggested otherwise and that mathematics textbooks reflect this fact. It is only because the sum rule's failure in the infinite case is being contrasted with its holding in all finite cases that the non-proof involving infinitely many δ_i derives explanatory power by showing where this contrast comes from. Because this contrast informs the why question, the kind of proof that an explanation identifies as failing to generalize to prove the sum rule in the infinite case must be a kind of proof by which the sum rule can be proved in finite cases.

This lesson takes us to the second obstacle to expanding Steiner's account to cover the explanatory non-proof in the previous section. The obstacle is that when we consider various proofs of the sum rule in finite cases (that fail to generalize to prove the sum rule in the infinite case—an unnecessary qualification, since the sum rule fails in the infinite case and so cannot be proved there), we find that not all of these proofs explain why the sum rule fails in the infinite case. What gave the argument in the previous section (involving infinitely many δ_i) its power to explain why the sum rule fails in the infinite case is not merely the feature that the Steineresque account captures: that the argument revealed a certain proof idea to be non-generalizable to the infinite case, despite its proving the sum rule in finite cases. Rather, the non-proof in the previous section derives its explanatory power from the *particular way* in which it shows the proof idea to be non-generalizable to the infinite case.

What particular way was that? Let me put it crudely for the moment. In Sect. 2, we saw that the proof idea by which we proved the sum rule for two functions, for three functions, and so forth fails to carry over to the infinite case. Crucially, we saw something more: we identified what stops the proof idea from generalizing and thereby prevents the sum rule from holding in the infinite case. Our seeing what stops the proof idea from generalizing involves something more than our discovering merely that those proofs regarding finite cases fail to generalize to the infinite case. The way they fail to generalize reveals the mechanism behind this failure—the cause of the problem. Of course, in mathematics we are not dealing with causal mechanisms![8] Nevertheless, the Steineresque account does not capture the fact that the explanation works by identifying something as what's *keeping* the proof idea that succeeded in the finite cases from also succeeding in the infinite case.

Less picturesquely, the explanation points out that what's holding the proof idea back from applying in the infinite case is that with infinitely many functions and hence infinitely many δ_i, there is no guarantee that some positive number is less than or equal to every δ_i. This consideration prevents the proof from generalizing to the infinite case and thereby accounts for the sum rule's failure there. The role played by this consideration in supplying the argument with explanatory power must be captured by an adequate philosophical account of mathematical non-proofs that explain.

To appreciate this point, let's look at an argument that satisfies the Steineresque account but intuitively does not explain why the sum rule fails in the infinite case. This argument will show another proof idea that succeeds in proving the sum rule in finite cases to be non-generalizable to the infinite case. But the argument will differ from the explanatory non-proof in the previous section (involving infinitely many δ_i) in that it will not reveal what stops the sum rule from holding in the infinite case. More precisely, the argument will fail to identify a further feature of the infinite case that contrasts with the finite cases and keeps the proof idea from carrying over to the infinite case. By a *further* respect in which the cases differ, I mean a respect beyond the fact that one case is infinite and the other cases are finite and also beyond the fact that the proof idea fails to apply to the infinite case though applying to the finite cases.

Before giving the argument that another proof idea fails to generalize to the infinite case, I want to say why it will turn out to be important whether or not an argument identifies a further feature by which the infinite case differs from the finite cases. In the following section, I will propose that for a non-proof of the sum rule's failure in the infinite case to explain why it fails there, the argument needs to identify a difference between the infinite and finite cases (other than that one is infinite and the others are finite) that is distinct from and so can account for the proof idea's failure to carry over to the infinite case. The difference identified by the explanatory non-proof (involving infinitely many δ_i) given in Sect. 2 concerned whether or not there must be some positive number less than or equal to every δ_i.

No analogous difference is identified by the following "bootstrapping" argument:

[8] Nevertheless, talk of "mechanism" is sometimes used by those who study math education in order to gesture toward the difference between a mathematical explanation of some fact and a proof of it that does not explain why it holds. For instance:

> Specific counter-examples are examples that merely satisfy the task of refuting a statement, and do not contribute to the understanding of the general case …. On the other hand, general examples uncover the crucial mechanism involved in the situation. This mechanism is both an explanation to the fact that the claim can be refuted, as well as a manifestation that counter-examples can be generated. (Peled and Zaslavsky 1997: 58–59, cf. 50)

This point applies to the explanation of the sum rule's failing in the infinite case and how this explanation differs in explanatory power from the specific counterexample (also given in Sect. 2) to the sum rule for the infinite case.

> Begin with any proof (such as the one given in the previous section) for the sum rule for two functions: $(f + g)'(x) = f'(x) + g'(x)$. Apply the sum rule just proved to the two functions $(f + g)(x)$ and $h(x)$, showing that $(f + g + h)'(x) = (f + g)'(x) + h'(x)$. Use $(f + g)'(x) = f'(x) + g'(x)$ to substitute for $(f + g)'(x)$, thereby proving $(f + g + h)'(x) = f'(x) + g'(x) + h'(x)$, i.e., the sum rule for three functions. In the same way apply the sum rule for two functions to the functions $(f + g + h)(x)$ and $j(x)$ to show $(f + g + h + j)'(x) = (f + g + h)'(x) + j'(x)$. Use the just proved $(f + g + h)'(x) = f'(x) + g'(x) + h'(x)$ to substitute for $(f + g + h)'(x)$, thereby proving the sum rule for four functions. And so on for five functions, six functions, ….[9]

This argument is non-generalizable to the infinite case; it can reach only finite numbers of functions.

Like the explanatory non-proof (involving infinitely many δ_i) given in Sect. 2, this argument does not prove that the sum rule fails in the infinite case. The Steineresque proposal regards this argument as explaining why the sum rule does not apply to the infinite case. Intuitively, however, this argument fails to supply an explanation (and I have never seen it cited in the mathematics literature as explanatory). Unlike the explanatory non-proof given in Sect. 2, this argument does not identify something as blocking the argument from extending to the infinite case. The proof idea (roughly, to bootstrap from two functions to three to four to …) simply lacks the potential ever to get to the infinite case. Nothing needed to step in and derail it before it reached the infinite case. It was never going to get that far.[10]

Hence, that some proof idea generalizes to all finite cases, but fails to prove the sum rule in the infinite case, may nevertheless not explain why that the sum rule fails in the infinite case. In order to explain, the failure to generalize must be shown to arise from the infinite case's failure to share some feature (besides finitude) with all of the finite cases. In identifying that feature, an explanatory non-proof specifies why the proof in finite cases gets derailed when we try to generalize it to the infinite case—and thereby explains why the sum rule fails in the infinite case. Let's now look at an account of mathematical explanation that captures this aspect of our example and so is sensitive not only to a proof idea's non-generalizability, but also to the way in which it fails to generalize.

4 My Proposal

With regard to the sum rule's failure in the infinite case, what must an account of mathematical explanations do? It must account not only for the explanatory power of the non-proof given in Sect. 2, but also for the explanatory impotence of the

[9] This argument could equally well have been expressed in terms of mathematical induction.

[10] The principle of mathematical induction (see previous footnote) does not allow the argument to extend to the infinite case, but does not specify any difference between the finite and infinite cases (other than that the latter is infinite) that is standing in the way. The infinite case simply fails to fall within the principle's scope.

"proof by example" in Sect. 2 and of the bootstrapping argument in the previous section. We also expect an account of explanatory non-proofs to cohere with an account of mathematical explanations that prove what they explain. I presume (as does Steiner) that these kinds of explanation work in similar ways—just as philosophers who work on scientific explanation widely presume that explanations where the explanandum does not follow logically from the explanans, such as statistical explanations, work in much (though, of course, not entirely) the same way as explanations in which the explanans entails the explanandum. Accordingly, I'll start by sketching an account of explanatory proofs and then show how that account expands naturally to cover explanatory non-proofs. I will incorporate into my account the lessons we learned from the Steineresque proposal's deficiencies.

Let's return to the mathematical explanation of the calculator-number theorem, which I introduced at the start of Sect. 1. What's striking about this theorem is that it reveals a respect in which all of the calculator numbers are alike (namely, they are all divisible by 37). The proof of this theorem that explains why it holds exploits another respect in which the calculator numbers are all alike in virtue of being calculator numbers—a respect that is distinct from (though required by) their being calculator numbers (and that is also distinct from their being divisible by 37). This similarity is that each calculator number can be expressed as $10^5\text{a} + 10^4(\text{a} + \text{d}) + 10^3(\text{a} + 2\text{d}) + 10^2(\text{a} + 2\text{d}) + 10(\text{a} + \text{d}) + \text{a}$ where a, a + d, a + 2d are three integers in arithmetic progression (these three integers can be the three digits on the calculator keypads that, taken forwards and backwards, generate the given calculator number). The explanatory proof thus traces the fact that every calculator number is divisible by 37 to another property that they have in common by virtue of being calculator numbers. In this respect, the explanatory proof differs sharply from the non-explanatory, brute-force proof that checks each of the sixteen calculator numbers individually. That proof identifies no common origin of the calculator numbers' common divisibility by 37.

This difference between the explanatory and non-explanatory proofs of the calculator-number theorem suggests that a proof explains a theorem concerning a given set up if and only if that theorem has some salient feature and the proof exploits a feature of the same kind that is possessed by the set up. The relevant kind of feature in the calculator-number example is a similarity among all of the calculator numbers. But an entirely different kind of feature might be salient in another mathematical result. For example, the result being explained might strikingly exhibit a certain symmetry. A proof of that result explains why it holds if and only if that proof derives the result by exploiting a similar symmetry in the set up. On this view, the distinction between proofs that explain *why* some theorem is true and proofs that merely establish *that* the theorem is true exists only in a context where some feature of the theorem is salient. This proposal correctly predicts an observation we made earlier: for a result having no especially striking feature, such as that $\int_1^3 (x^3 - 5x + 2)dx = 4$, there is nothing that it would be to explain why it holds; there is only a proof that it holds. Likewise, with regard to the theorem that the sum rule fails in the infinite case, we saw that there is nothing it would be to explain why this theorem holds until the sum rule's failure in the infinite case is contrasted with its holding in all finite cases. Where that contrast is striking, it informs what an explanation would be.

Elsewhere (2014, 2016) I have elaborated and defended this account of the difference between explanatory and non-explanatory proofs. Without pausing now to do that, let's see how this rough proposal expands naturally to cover an explanatory non-proof. Whereas the salient feature of the calculator-number theorem is the *similarity* it reveals among all of the calculator numbers, the salient feature of the sum-rule theorem regarding the infinite case is its *difference* from the sum-rule theorem for finite cases. Hence, just as a proof explains the calculator-number theorem by taking the salient similarity that the theorem ascribes to the calculator numbers and deriving it from another respect in which the calculator numbers are alike, so an argument explains the sum rule's failure in the infinite case by tracing the salient difference between the infinite and finite cases to another respect in which they differ—a respect that is distinct from (though required by) the fact that one is infinite and the other finite (and that is also distinct from the sum rule's holding of one but not the other). To trace the salient difference to this other difference, it suffices to show that this other difference keeps a proof idea that works for the finite case from generalizing to the infinite case. As we have seen, the difference that the explanatory non-proof points out is that for finitely many functions and hence finitely many δ_i, there exists some positive number less than or equal to every δ_i, but there need not exist such a positive number for infinitely many functions and hence infinitely many δ_i. The proof regarding finite cases appeals to this guarantee, so without it in the infinite case, the proof of the sum rule in the finite case cannot generalize to the infinite case.

On my proposal, this explanatory non-proof explains in much the same way as the proof explaining the calculator-number theorem. A certain similarity among the calculator numbers (that they are all expressible in the same form) allows the same proof to show them all to be divisible by 37, and when the salient feature of the calculator-number theorem is the similarity that it reveals among the calculator numbers, then a proof of the theorem explains why it holds if the proof exploits another similarity they bear (distinct from but required by their all being calculator numbers). Just as the explanandum in the calculator-number example makes salient a similarity among the calculator numbers, so the sum rule's failure in the infinite case makes salient a difference between the infinite and finite cases. Just as the explanation of the calculator-number theorem works by tracing the salient similarity to another similarity among the calculator numbers, so the explanation of the sum-rule theorem works by tracing the salient difference that the theorem reveals between infinite and finite cases to another difference between them (distinct from but required by their difference in cardinality).

My proposal thus gives an account of this non-proof's explanatory power that coheres with its account of explanatory proofs. My proposal also incorporates one of the lessons we learned from examining the Steineresque proposal: that a proof idea's failure to yield a proof that the sum rule holds in the infinite case explains the sum rule's failure in the infinite case only if that proof idea enables the sum rule to be proved in finite cases. Only such a proof idea allows us to take the difference between the sum rule in the infinite and finite cases and trace it to some other difference between those cases.

Furthermore, my account vindicates our earlier thought that the explanatory non-proof derives its explanatory power from revealing the "mechanism" responsible for the sum rule's failure in the infinite case. Let's see how this result arises from the fact that on my account, the explanation works by taking the difference between the sum rule in the infinite and finite cases and tracing this difference back to some other difference between those cases that is distinct from but required by their difference in cardinality. In so tracing back the difference in the sum rule's holding, the explanatory non-proof identifies what it is about being an infinite case, by contrast with being a finite case, that blocks a proof for the finite case from extending to the infinite case. The explanatory non-proof identifies the difference that makes this difference to be that finite and infinite cases differ in whether there is guaranteed to be some positive number less than or equal to every δ_i. In tracing the difference in the sum rule's holding back to this other difference, the explanatory non-proof reveals how this other difference makes a difference to the sum rule's holding; that is, it points out which step of the proof regarding finite cases is blocked by this feature of infinite cases. In identifying what it is about being an infinite case that stops the proof from going through, as well as how this feature of infinite cases stops the proof, the explanation gives the "mechanism" by which the proof in the finite case is kept from extending to the infinite case.

One way to identify what blocks the proof's extension to the infinite case (thereby revealing the "mechanism" at work) is to specify what would avoid this blockage—that is, to specify what some infinite case would have to be like in order for the proof in the finite case to carry over to that infinite case (namely, for there to be some positive number less than or equal to every δ_i). Some infinite cases allow this to happen (see footnote 6). By specifying what infinite cases would have to be like in order for the proof in the finite case to carry over to them, the explanatory non-proof may be helpful in constructing additional infinite cases where the sum rule fails—more helpful than either the "proof by example" or the bootstrapping proof. But I do not take the explanatory non-proof's helpfulness in this regard to be the source of its explanatory power. On my view, it is the other way around. In other words, the explanatory non-proof's power to explain why the sum rule fails in the infinite case is what makes the explanatory non-proof helpful in constructing counterexamples to the sum rule in the infinite case, rather than its helpfulness in constructing counterexamples making it explanatory.

By contrast with the explanatory non-proof, the bootstrapping proof's non-generalizability does not specify a "mechanism" responsible for the sum rule's failure; the bootstrapping proof fails to indicate what infinite cases must be like in order for a proof of the sum rule to carry over to them. We saw how the explanatory non-proof points out a difference between finite and infinite sums that makes a difference to whether a certain strategy for proving the sum rule succeeds (namely, that in the finite but not the infinite case, there is guaranteed to be a positive number less than or equal to every δ_i). No such difference between finite and infinite sums is identified as responsible for the bootstrapping proof's failure to reach the infinite case. Although the bootstrapping proof never manages to reach the infinite case, the proof contains no particular step that goes through in the finite case but not when we try to generalize the proof to the infinite case. In this way, my account says why the bootstrapping proof of the finite sum-rule theorem, despite not generalizing to the infinite case, fails to explain the sum rule's breaking down for the infinite case. As we saw, the Steineresque account could not do this. My account incorporates the lesson we learned from the second obstacle encountered by the Steineresque account: that the particular way in which a proof fails to generalize to the infinite case, despite proving the sum rule for finite cases, makes a difference to whether its non-generalizability can explain.[11]

Furthermore, my proposal explains why we do not need to give a proof of the sum-rule theorem in the infinite case in order to explain why that theorem holds. When the salient feature of the sum-rule theorem for the infinite case is its *difference* from the sum-rule theorem for finite cases, its explanation need only find another difference between the infinite and finite cases and show how that difference turns out to make a difference to the sum rule's holding. A respect in which the infinite case differs from finite cases can make a relevant difference merely by derailing in the infinite case a proof idea that works to prove the sum rule in the finite case. Stopping the proof is making a difference.[12]

By contrast, the Steineresque account cannot say why we do not need to give a proof of the theorem in order to explain it. The explanandum is a theorem, so why should anything less than a proof of it be enough to answer the why question? The Steineresque account cannot say why, because it does not recognize that what is

[11] As we saw at the very start of the paper, mathematicians regard the fact that two calculator numbers are both divisible by 37 as no coincidence, and that is because of the proof that explains why every calculator number is divisible by 37. I have suggested that when a result's salient feature is that it identifies a respect in which various cases are alike, then a proof explains that result exactly when the proof exploits some other respect in which those cases are alike and from there arrives at the result by treating all of the cases in the same way. I now suggest that if the result with such a salient feature has no such explanation, then it is a mathematical coincidence. Likewise, suppose we had not yet proved that the sum rule holds of every pair of differentiable functions and suppose we found that for four particular functions $f_1, \ldots, f_4$, $f_1'(x) + f_2'(x) = (f_1 + f_2)'(x)$ and $f_3'(x) + f_4'(x) = (f_3 + f_4)'(x)$. We might then wonder whether or not this fact is a coincidence. The general proof of the sum rule for two functions makes it no coincidence. However, let's add to those two facts an example where the sum rule holds for the sum of infinitely many functions. (There are such examples; see footnote 6.) That the sum rule holds for those three cases is a coincidence since the sum rule's holding of those three cases cannot be derived from the respects in which those three cases are alike (e.g., that each is the sum of differentiable functions on the real numbers).

[12] This is how I would reply to the objection that the explanatory non-proof I have been examining does not explain why the sum rule fails in the infinite case; it explains merely why a given proof in the finite case does not carry over to the infinite case. I agree that it explains why the proof does not carry over. But when the salient feature of the sum rule's failure in the infinite case is that by this failure, the infinite case stands in contrast to all finite cases, then we can explain the sum rule's failure in the infinite case by identifying another difference between finite and infinite cases that stops a proof of the sum rule in the finite case from generalizing to the infinite case. In this respect, the example is analogous to a standard case from the literature on scientific explanation. Suppose tertiary syphilis is necessary for the development of paresis (though very few patients with tertiary syphilis develop paresis). Suppose we ask why Jones developed paresis—and we pose this question in a context where the salient contrast is with Smith, who did not develop paresis. That Jones had tertiary syphilis (where Smith did not) answers the question; it explains more than merely why the mechanism that kept Smith from getting paresis failed to carry over to Jones.

required to explain some result depends on what feature of the explanandum is salient—so that if its salient feature is its *difference* from another result, then its explanation need not prove it. In the context of the sum rule for finite cases, to ask for an explanation of the sum-rule theorem in the infinite case is to ask where its difference from the theorem for finite cases comes from. We can reveal a difference that the sum's cardinality makes to the sum rule without proving the sum-rule theorem in the infinite case: by revealing precisely how the cardinality makes a difference to whether a proof goes through. Conversely, the "proof by example" in Sect. 2 fails to specify where the infinite sum-rule theorem's difference from the finite sum-rule theorem comes from, so the proof by example fails to explain the infinite sum-rule theorem—despite proving it.

5 Conclusion

Though we have focused on a single example of an explanatory non-proof, many other explanations in mathematics work in the very same way. For instance, suppose we ask why it is *not* the case that for every continuous function on the *rational* numbers in the interval [a, b] where $f(a) < 0 < f(b)$, there is a $c \in [a, b]$ such that $f(c) = 0$. With this why question put so baldly, it seems to me indeterminate what it would take to explain why this fact holds if this task is supposed to be distinct from proving that it holds.

However, suppose we mention the explanandum along with the intermediate-value theorem (IVT): that for every continuous function on the *real* numbers in the interval [a, b] where $f(a) < 0 < f(b)$, there *is* a $c \in [a, b]$ such that $f(c) = 0$. In this context, the explanandum's salient feature is its contrast with the IVT, so to ask why the explanandum holds is to ask why the IVT's analogue for the rationals (i.e., the explanandum's negation) fails to hold. This why question is not answered by a "proof by example" that the IVT's analogue does not apply to the rationals; for instance, the why question is not answered by noting that for the function $f(x) = x^2 - 2$ defined for rational x's, $f(0) = -2$ and $f(2) = 2$ but there is no rational number x such that $f(x) = 0$ (since $\sqrt{2}$ is irrational).

An explanation is instead supplied by a proof of the IVT that at a certain step exploits a property of the reals that is not possessed by the rationals. Such an explanation identifies a specific difference between the reals and rationals that is responsible for their difference with respect to the proof idea's success in proving an IVT-type theorem. Typically, such a proof of the IVT exploits the real numbers' possession of the least-upper-bound property: that every nonempty set of real numbers having an upper bound is guaranteed to have a least upper bound. The rationals lack this property (for instance, the set of rationals having their square less than 2 has an upper bound, such as 5, but no least upper bound since for any upper bound among the rationals, there is a smaller one). The rationals' failure to possess the least-upper-bound property blocks the standard proof of the IVT from generalizing to prove the IVT's analogue for the rationals.[13] This fact fails to prove that the IVT-analogue for the rationals is violated but explains why it is in fact violated. This explanatory non-proof works by tracing the salient difference between the rationals and reals to another respect in which they differ—a respect that is distinct from (though required by) the fact that one is the set of reals and the other is the set of rationals.[14] In all of these respects, this example of an explanatory non-proof is like the one that we have been examining.

Closely related are explanations that answer question of the form "Why is this result so hard to prove?" and "Why is this problem so much harder to solve than that one, when they appear so much alike?" When two problems are being contrasted as to their difficulty, the answer to the why question must trace the difference in their difficulty to some other difference between them that prevents a means of solving one of them from generalizing to solve the other. Since the explanandum is not a theorem, its explanation cannot be a proof since only theorems have proofs.

Questions of the form "Why is this result so hard to prove (when that one isn't)?" have long been a focus of mathematical research. Here is one such famous question with which mathematicians remain engaged today. Standard techniques allow all of the following results to be derived:

$$1-1/3+1/5-1/7+\cdots=\pi/4$$
$$1/1^2+1/2^2+1/3^2+\cdots=\pi^2/6$$
$$1/1^2-1/2^2+1/3^2-1/4^2+\cdots=\pi^2/12$$
$$1/1^3-1/3^3+1/5^3-1/7^3+\cdots=\pi^3/32$$

But despite its similarity to all of these examples, no one has found such an expression to complete the equation

$$1/1^3+1/2^3+1/3^3+\cdots=?$$

Why is this sum[15] so much more difficult than the others, from which it differs only slightly? This is a notorious why question:

> Why the problem goes from "not hard" to "stupendously hard" just by making what appears to be such minor changes remains an immensely deep mystery (Nahin 2009: 99).

Philosophers should devote some attention to understanding how mathematical explanations that answer "Why is this problem so difficult?" questions operate.[16]

[13] Resnik and Kushner (1987: 147–152) use this example to argue that Steiner's requirement that an explanatory proof generalize is too high since this proof of the IVT fails to generalize to the rationals.

[14] Many textbooks (e.g., Spivak 1980: 121–122) prove the IVT by appealing to the least-upper-bound property and show how this proof "goes wrong" (Spivak 1980: 114) for the rationals.

[15] This sum is $\zeta(3)$, where $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta function. $\zeta(s)$ is intractable for any positive odd integer (other than $s = 1$, for which the sum diverges).

[16] As another example where a mathematician explains why one problem turns out to be so much more difficult than another, seemingly closely related one, consider these remarks that Hardy wrote in his copy

Presumably, to explain why a given problem is so difficult, we must identify the obstacle that blocks some relevant standard proof strategy from succeeding—just as we did in explaining why the sum rule fails in the infinite case.

Explanatory non-proofs have been understudied. It is important that philosophers interested in mathematical explanation not confine their attention to mathematical proofs. An account of the distinction between proofs that do and proofs that do not explain why the theorems they prove hold should extend in some natural way to cover explanatory non-proofs.

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Footnote 16 continued

of a letter he had written to Ramanujan: "When r is even $f(xe^{-p\pi i/q})$ is an *elementary* function. The same sort of method applies but is *much* easier. Hence we see *why* the odd case is so much harder" (Berndt and Rankin 1995: 152).