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Dimensional Explanations

Marc Lange
The University of North Carolina at Chapel Hill

1. Introduction

Consider a small stone tied to a string and whirled around at a constant speed (figure 1). Why, one might ask, is the tension in the string proportional to the *square* of the stone's angular speed (rather than, say, varying linearly with its speed, or varying with the cube of its speed)? This is a why-question—a request for a scientific explanation.

Here, perhaps, is an answer to the why-question. To a sufficiently good approximation (for example, when the string's mass and the stone's size are negligible), the tension F depends on no more than the following three quantities: (i) the stone's mass m, (ii) its angular speed ω, and (iii) the string's length r. These three quantities suffice to physically characterize the system. In terms of the dimensions of length (L), mass (M), and time (T), F's dimensions are LMT^{-2}, m's dimension is M, ω's dimension is T^{-1}, and r's dimension is L. In tabular form,

	F	m	ω	r
L	1	0	0	1
M	1	1	0	0
T	−2	0	−1	0

If (to a sufficiently good approximation) F is proportional to $m^\alpha\ \omega^\beta\ r^\gamma$, then the table gives us three simple simultaneous equations (one from each line) with three unknowns:

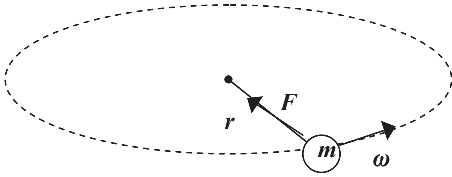

Figure 1

$$\text{From L:} \quad 1 = \gamma$$

$$\text{From M:} \quad 1 = \alpha$$

$$\text{From T:} \quad -2 = -\beta$$

Since $\alpha = 1$, $\beta = 2$, and $\gamma = 1$, F is proportional to $m\omega^2 r$.

Apparently, then, dimensional considerations suffice to explain the explanandum: F must be proportional to ω^2, rather than to ω or to ω^3, because F is a function of m, ω, and r alone, and among these three quantities, ω is the only one capable of supplying F's time dimension (T^{-2}). If such an argument amounts to a "dimensional explanation", then this variety of explanation has been unjustly neglected in the enormous philosophical literature on scientific explanation.

Of course, dimensional analysis is well-known in physics and engineering as a shortcut (Birkhoff 1950, Bridgman 1931, Langhaar 1951, Sedov 1959). If you have identified all of the relevant quantities characterizing a given physical system, including the dimensional constants (such as the speed of light c and Newton's gravitational-force constant G), and if you know their dimensions, then by dimensional considerations alone, you may learn a considerable amount about the relation holding among those quantities—as we just did regarding the string tension's relation to m, ω, and r. A physics student who has forgotten whether the tension is proportional to ω, ω^2, or ω^3 could figure it out on purely dimensional grounds (as long as she remembers which quantities are relevant to F and their dimensions).

However, I have rarely seen dimensional arguments characterized as possessing explanatory power.[1] Yet they appear to do so. These putative dimensional explanations form the subject of this paper.

Consider a derivative law[2] (such as the example we just saw: that the centripetal force on a small body of mass m moving uniformly with angular speed ω in a circular orbit of radius r is proportional to $m\omega^2 r$). This law follows from various, more fundamental laws. However, I shall argue, a derivative law's dimensional explanation can supply a kind of understanding that is not provided by its derivation from more fundamental laws. For instance, its dimensional explanation may reveal which features of the more fundamental laws entailing it are in fact responsible for it and which features are explanatorily irrelevant to it. Likewise, different features of the same derivative law may receive quite different dimensional explanations, whereas they are not traced to different sources in the law's derivation from more fundamental laws. A dimensional explanation may reveal some features of a given derivative law to be independent of some features of the more fundamental laws entailing it.

A derivative law concerning one physical system may turn out to be similar in form to a derivative law concerning a physically unrelated system. A separate derivation of each derivative law from various, more fundamental laws may explain each. However, neither one of the derivative laws explains the other, and the separate derivations may fail to identify any feature common to the two systems as responsible for the similarity of the two derivative laws concerning them. The pair of explanations may thus portray the similarity between the laws as a kind of "coincidence" (albeit a naturally necessary one). On the other hand, the two systems may be *dimensionally* similar in certain respects. The dimensional features common to the two systems may account for the similarity in the derivative laws concerning them. In that event, a dimensional explanation succeeds in unifying what separate derivations from more fundamental laws fail to unify. The dimensional explanation then correctly characterizes the derivative laws' similarity as no coincidence, but rather as explained by the dimensional architecture common to the two systems.[3]

Likewise, a dimensional explanation may point to certain dimensional differences between two physical systems as responsible for certain differences between the derivative laws concerning each of them. No such explanations of those differences are supplied by separate derivations of the two derivative laws from various, more fundamental laws.

Dimensional thinking not only yields new explanations of antecedently appreciated phenomena, but also (I shall argue) identifies new phenomena to explain—phenomena that can be expressed only in dimensional terms. Furthermore, having argued that dimensional explanations can reveal a given derivative law to be independent of certain features of the more fundamental laws entailing it, I will argue that in some dimensional explanations, part of what explains a derivative law is precisely its independence from certain features of those more fundamental laws. Finally, I will suggest that some dimensional explanations may proceed from "meta-laws" that impose various constraints on first-order laws (just as symmetry principles in physics are usually taken as constraining force laws and other first-order laws).

Dimensional explanation may not only constitute an important and overlooked variety of scientific explanation, but also help us to understand how a derivative law's scientific explanation differs from its mere deduction from more fundamental laws.

2. A Dimensional Explanation of a Derivative Law May Reveal Certain Features of the More Fundamental Laws Entailing It to be Explanatorily Irrelevant to It

Consider a planet of mass m orbiting a star of mass M, feeling (to a sufficiently good approximation) only the star's gravitational influence and orbiting with period T in a circular orbit of radius r. Let's focus solely on T's relation to r: that $T \propto r^{3/2}$. (The symbol "$\propto$" means "is proportional to".) We shall now compare two possible explanations of this proportionality.

Our first candidate explanation derives the explanandum from more fundamental laws: Newton's laws of motion and gravity.[4] These laws tell us that that $F = ma$ and $F = GMm/r^2$, where F is the force on the planet and a is the planet's acceleration. A body undergoing circular motion at a constant speed v experiences an acceleration a of v^2/r towards the center. Hence,

$$GMm/r^2 = mv^2/r,$$

and so

$$GM/r^2 = v^2/r,$$

and therefore

$$GM/r = v^2.$$

For a circular orbit with circumference $c = 2\pi r$, the period equals the distance c covered in one revolution divided by the speed v. That is,

$$T = 2\pi r/v,$$

and so

$$T^2 = 4\pi^2 r^2/v^2.$$

By inserting the expression for v^2 derived from Newton's laws, we find

$$T^2 = 4\pi^2 r^2/(GM/r) = 4\pi^2 r^3/GM.$$

Perhaps we have thereby explained why $T \propto r^{3/2}$.

Now I shall work towards giving a "dimensional explanation" of the fact that $T \propto r^{3/2}$. The explanans will be that T stands in a "dimensionally homogeneous" relation to some subset of m, M, G, and r. A relation is "dimensionally homogeneous" if and only if the relation holds in any system of units for the various fundamental dimensions of the quantities so related.[5] Of course, *r*'s numerical value in meters will differ from its numerical value in feet. But the same *relation* can hold among the quantities' numerical values no matter what units are used to measure those quantities, even if the numerical values so related differ in different units. I'll say a bit more about dimensional homogeneity (and the explanans in a dimensional explanation) in sections 6 and 7.

The concept of dimensional homogeneity presupposes the concept of two units (e.g., grams and slugs) counting as different ways of specifying the same quantity (e.g., an object's mass). Two units so qualify only if necessarily whenever x is the numerical value in one unit and y is the corresponding numerical value in the other unit, $y = cx$ for some constant $c > 0$—the "conversion factor" between the units (Bridgman 1931: 18–21). This condition is motivated by the idea that two units are not means of specifying the same quantity if, by changing between the units, the ratio between two measurements is not preserved.[6] For example, since grams and slugs are units for expressing the same quantity (mass), my measure in grams is twice my son's measure in grams if and only if my measure in slugs is twice my son's measure in slugs.[7]

Suppose $v = \mathrm{f}(s, t, u, \ldots)$ is continuous and dimensionally homogeneous, where $s, t, u, v, \ldots$ are the (positive-valued) quantities expressed in one system of units. Suppose we now use a new system of units for these quantities, involving positive-valued conversion factors $c, d, e, \ldots$, respectively; s when converted into the new units becomes cs, t becomes dt, u becomes eu, and so forth. Then since the equation is dimensionally homogeneous, $\mathrm{f}(cs, dt, eu, \ldots)$ must equal whatever v becomes when converted into the new system of units, which must be v multiplied by some function ϕ of $c, d, e, \ldots$ (which gives the conversion factor for the quantity expressed by v). That is, $\mathrm{f}(cs, dt, eu, \ldots) = \phi(c, d, e \ldots)\, \mathrm{f}(s, t, u, \ldots)$. It can be shown (see, e.g., Bridgman 1931: 21–2, Birkhoff 1950: 87–8, Luce 1959: 87, Ellis 1966: 204) that this condition holds for any new system of units if and only if $\mathrm{f}(s, t, u, \ldots)$ is proportional to $s^\alpha\, t^\beta\, u^\gamma \ldots$, where the constant of proportionality and the exponents are dimensionless constants—as long as there is no dimensionless combination of $s, t, u, \ldots$. (If there is, then $\mathrm{f}(s, t, u, \ldots)$ is proportional to $s^\alpha\, t^\beta\, u^\gamma \ldots$ times some function of the dimensionless combination(s). Dimensional analysis alone is insufficient to identify the function.)

Let us return to the dimensional explanation of the fact that $T \propto r^{3/2}$. The explanans was that T stands in some "dimensionally homogeneous" relation to (some subset of) m, M, G, and r. As we have just seen, this entails that T is proportional to $m^\alpha\, M^\beta\, G^\gamma\, r^\delta$ (times some function of M/m). If distance (L), mass (M), and time (T) are taken as the fundamental dimensions, then our table is as follows:

	T	m	M	G	r
L	0	0	0	3	1
M	0	1	1	−1	0
T	1	0	0	−2	0

The table gives us three simultaneous equations (one from each line) with four unknowns:

$$\text{From L:} \quad 0 = 3\gamma + \delta$$

$$\text{From M:} \quad 0 = \alpha + \beta - \gamma$$

$$\text{From T:} \quad 1 = -2\gamma$$

From the L and T equations, it follows that $\gamma = -{}^1\!/_2$ and $\delta = 3/2$. So T is proportional to $r^{3/2}$, which is what we were trying to explain.

How does this "explanation" relate to the derivation (given earlier) of the $T \propto r^{3/2}$ law from more fundamental laws? One possibility is that the dimensional argument is not explanatory: although $T \propto r^{3/2}$ is *entailed* by the fact that T stands in a dimensionally homogeneous relation to (some subset of) m, M, G, and r, both of these facts are explained by $F = ma$ and $F = GMm/r^2$. The supposed explanans and explanandum of a "dimensional explanation" have a common origin; one is not responsible for the other.

But dimensional arguments appear explanatory. Why might one insist that nevertheless, they are not? Perhaps because a dimensional argument cannot yield more than various proportionalities—for instance, that $T \propto r^{3/2}$ and $T \propto G^{-1/2}$, whereas more fundamental laws entail the complete equation ($T = 2\pi\sqrt{(r^3/GM)}$), including the values of dimensionless constants of proportionality (such as the 2π in this case) and that T is independent of the planet's mass (m).[8] This difference might seem to suggest that whereas the derivation from more fundamental laws is explanatory, the dimensional argument is not. However, I see no reason why the dimensional argument would have to explain the entire equation above in order to explain why $T \propto r^{3/2}$ and $T \propto G^{-1/2}$.

I shall suggest that the dimensional argument *is* explanatory. Indeed, it reveals that as far as explaining $T \propto r^{3/2}$ is concerned, certain features of the more fundamental laws are idle. For example, $T \propto r^{3/2}$ depends neither on G's specific value nor on those features of $F = ma$ and $F = GMm/r^2$ captured by the M equation. Rather, $T \propto r^{3/2}$ arises from G's being the only source of T dimensions from among m, M, G, and r, and r's being the only remaining source of L dimensions to compensate for the L dimensions from G. The M equation does not figure in the argument. The explanans is then merely that T stands in a relation to (some subset of) m, M, G, and r—a relation that is dimensionally homogeneous in terms of L and T. This explanans captures the only feature of the fundamental laws that is explanatorily relevant to $T \propto r^{3/2}$.

On this view, $T \propto r^{3/2}$'s derivation from $F = ma$ and $F = GMm/r^2$ contains elements that are otiose as far as explaining $T \propto r^{3/2}$ is concerned. The derivation is therefore like the famous counterexample from Kyburg (1965) to Hempel and Oppenheim's D-N model: we cannot explain why a given sample dissolved by saying that it was table salt, hexed (i.e., a person wearing a funny hat waved a wand and mumbled something over it), and placed in water, and that as a matter of natural necessity, all hexed samples of table salt dissolve when placed in water.[9] To illustrate the independence of $T \propto r^{3/2}$ from the features of $F = ma$ and $F = GMm/r^2$ expressed by the M equation, let us suppose that (contrary to natural law) the gravitational force had been proportional to M^2m^2 rather than to Mm. Then although G's M dimension would have been different, its L and T dimensions would not. Thus, the L and T lines in the table would have been no different, and so $T \propto r^{3/2}$ would still have held.

The derivation from $F = ma$ and $F = GMm/r^2$ cannot reflect this independence, since there is no point in the course of that derivation after which the M dimensions of various quantities remain isolated from all of their other dimensions. The derivation we saw earlier begins with the actual, more fundamental laws, not some generalizations thereof, and it simply churns on, deducing consequences of those laws until it generates the explanandum. In contrast, the dimensional explanation begins by effectively teasing apart various features of those more fundamental laws, and then it keeps these strands apart, allowing the significance of each of them to be traced separately. Thus it can depict certain of these features as explanatorily relevant and others as making no contribution to the explanation.[10]

On this view, the dimensional argument explains why $T \propto r^{3/2}$ and the derivation from $F = ma$ and $F = GMm/r^2$ does not; it includes some facts that are explanatorily irrelevant to $T \propto r^{3/2}$. Another way to put this point involves turning to the explanans in the dimensional explanation and asking why *it* holds. Suppose the reason that T stands in a relation to (some subset of) m, M, G, and r that is dimensionally homogeneous in terms of L and T is because of laws such as $F = ma$ and $F = GMm/r^2$ in all of their detail. In that case, all of those details would be explanatorily relevant to $T \propto r^{3/2}$. But as we have just seen, certain features of the fundamental laws are not reflected in the dimensional explanans and so do not help to explain why $T \propto r^{3/2}$. The dimensional explanans is not *explained by* the fundamental laws. Neither is explanatorily prior to the other. Rather, the dimensional explanans expresses exactly the fact about the fundamental laws that is responsible for $T \propto r^{3/2}$.

In subsequent sections, I shall offer additional reasons to regard a dimensional argument as explaining why some derivative law holds and as supplying understanding that cannot be supplied even in principle by that derivative law's deduction from more fundamental laws.

3. Different Features of a Derivative Law May Receive Different Dimensional Explanations

I have just argued that a dimensional explanation may reveal certain aspects of the more fundamental laws entailing a given derivative law to be explanatorily irrelevant to that law. That is because a dimensional explanation disregards certain aspects of the more fundamental laws (by considering only their dimensional architecture) and treats certain other aspects separately from one another (by tracing individually each of several dimensions). The way in which dimensional explanations tease apart the various dimensions has another consequence: Different aspects of the same derivative law may receive different dimensional explanations. This is another respect in which a law's dimensional explanation may differ from the law's deduction from more fundamental laws. Just as a dimensional explanation may distinguish the contributions (if any) made by different aspects of the more fundamental laws, so also it may give separate explanations to different aspects of the law being explained.

Here is an example. Consider a body with mass m lying on a smooth table and subject to the force of a spring (figure 2).

Suppose we grasp the body and pull, stretching the spring to a distance x_m beyond its equilibrium length, and then let go. The body then moves back and forth. Here are two why-questions concerning this case:

(1) Why is the body's period of oscillation T proportional to $\surd(m/k)$, where each spring has a characteristic constant k?

(2) Why is T not a function of x_m?

These two questions receive the same answer in terms of a deduction from more fundamental laws: Newton's second law of motion ($F = ma$) and Hooke's law (that the force F exerted by the spring on the body is towards

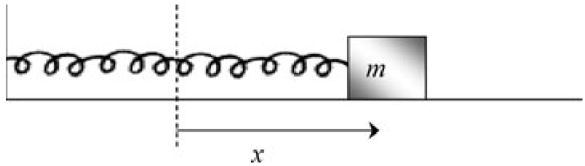

Figure 2

the spring's equilibrium position and, to a sufficiently good approximation, is proportional to the body's displacement x from that position). The answer to the two why-questions is that from these laws, we can deduce $x(t)$, the body's displacement as a function of time, which turns out to be an oscillation with a period $T = 2\pi\sqrt{(m/k)}$, so T is not a function of x_m and is proportional to $\sqrt{(m/k)}$. The derivation is as follows:

$$ma - F = 0$$

Let k be the constant of proportionality in Hooke's law,

$$F = -kx.$$

So

$$ma + kx = 0,$$

and so

$$a + (k/m)x = 0.$$

This differential equation (a is x's second-derivative with respect to time) is solved by

$$x(t) = x_m \cos(\sqrt{(k/m)}\, t + \phi),$$

which has a period of

$$2\pi\sqrt{(m/k)}.$$

Thus, as far as this deduction can reveal, the reason why T is not a function of x_m and is proportional to $\sqrt{(m/k)}$ is that—well, it just works out that way, considering Newton's second law and Hooke's law.

Let's compare this deduction to a dimensional explanation. The explanans is that (to a sufficiently good approximation) x_m, k, and m suffice to completely physically characterize the situation; they suffice (and may even be more than enough) to form a quantity standing in a dimensionally homogeneous relation to T. So here is our table:

	T	x_m	k	m
L	0	1	0	0
M	0	0	1	1
T	1	0	−2	0

If T is proportional to $x_m^{\alpha}\ k^{\beta}\ m^{\gamma}$, then it follows from the table's first line that $\alpha = 0$. In other words, the reason why T is not a function of x_m is that x_m is the only quantity characterizing the system that involves an L dimension, so there is nothing available to compensate for its L dimension in order to leave us with T (which has no L dimension).

This dimensional explanation of T's independence from x_m is distinct from the dimensional explanation of T's varying with $\sqrt{(m/k)}$. The latter explanation is given by the M and T equations:

$$\text{From M:} \quad 0 = \beta + \gamma$$

$$\text{From T:} \quad 1 = -2\beta$$

Solving, we find that $\beta = -^1/_2$ and $\gamma = ^1/_2$, and so that T is proportional to $k^{-^1/_2}\ m^{^1/_2}$, i.e., $\sqrt{(m/k)}$.

Different aspects of the derivative law thus receive different dimensional explanations, since those explanations pull the various dimensions apart. But those different aspects of the derivative law all arise together at the conclusion of the law's derivation from more fundamental laws.[11] The first dimensional explanation identifies the specific feature of the fundamental laws that is responsible for T's independence from x_m, and the second dimensional explanation distinguishes this feature from the one that is responsible for T's varying with $\sqrt{(m/k)}$. By contrast, these features are not distinguished in the derivation of $x(t) = x_m \cos(\sqrt{(k/m)}\ t + \phi)$.

4. Dimensional Explanations Can Unify what Derivations from More Fundamental Laws Cannot

In section 2, I argued that a dimensional explanation may reveal certain aspects of the more fundamental laws from which a given derivative law follows to be explanatorily irrelevant to that law. Hence, there should be pairs of laws that are entailed differently by more fundamental laws, but where those differences are confined to aspects of the more fundamental laws that turn out to be explanatorily irrelevant according to the dimensional explanations of the two derivative laws. In that event, the two derivative laws receive the same dimensional explanation. In other words, we should expect dimensional explanations to unify derivative laws that are derived separately from more fundamental laws.

For example, consider a wave of pressure propagating in a fluid (or elastic solid) surrounded by rigid walls (such as water in a rigid pipe). A compression moves through the fluid, so that at a given moment, there are alternating regions of compression and rarefaction. Over time, a small region of the fluid alternately undergoes compression and rarefaction, as the elements of the fluid in that region are pushed together or spread apart. Each element of the fluid oscillates back and forth along the same line as the wave propagates. That is, a pressure wave (such as a sound wave) is "longitudinal". Let us compare this wave to the wave propagating down a stretched string, such as a guitar string held taut by tuning pegs and plucked at one end. That wave is "transverse" in that the elements of the string oscillate back and forth in a direction perpendicular to the direction of the wave's propagation. (Assume that the amplitude of the wave is small compared to the length of the string.) Despite the differences between these two cases, the wave's speed v in the two cases is given by remarkably similar equations:

longitudinal:	transverse:
$v = \sqrt{(B/\rho)}$	$v = \sqrt{(F/\mu)}$
where ρ is the medium's density,	where μ is the string's linear density,
B is its bulk modulus[12]	F is its tension

The analogy between ρ and μ is evident; the only difference between them reflects the fact that the medium in the longitudinal case occupies a volume, whereas the medium in the transverse case is a string (one-dimensional). The analogy between B and F may be less evident. A medium's bulk modulus B reflects how much additional pressure must be exerted on the medium to reduce its volume by a certain fraction; if, by changing the pressure it feels by dP, one changes its volume by dV in total, or by (dV/V) per each unit of its volume V, then its bulk modulus is given by

$$B = -dP/(dV/V).$$

(The minus sign makes B a positive quantity, since if $dP > 0$, then $dV < 0$, considering that increased pressure brings decreased volume.) If the fluid's bulk modulus is larger, then greater additional pressure is needed to compress the fluid by a certain fraction. In this respect, the bulk modulus is like the string's tension; if the string is tighter, then greater force is needed to pluck it—that is, to make it bend (which requires lengthening it). In short, $\sqrt{(B/\rho)}$ and $\sqrt{(F/\mu)}$ are alike in that each takes the form $\sqrt{}$(medium's elasticity/density).

Why is there such a similarity between these physically dissimilar kinds of waves? A derivation of the transverse wave's v from more fundamental laws explains why that quantity equals $\sqrt{(F/\mu)}$. An unrelated derivation of the longitudinal wave's v from more fundamental laws explains why that quantity equals $\sqrt{(B/\rho)}$. Do these separate explanations for each wave's v explain the similarity in the two v equations—that is, do they together explain why the two waves' velocities are each proportional to $\sqrt{}$(medium's elasticity/density)? Only if this similarity in the waves' v is in fact coincidental. But it turns out not to be coincidental. Rather, as the dimensional explanation

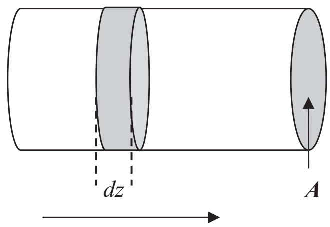

Figure 3

reveals, their similarity actually arises from a feature common to the two kinds of waves.

We can see this by starting with the longitudinal wave (figure 3). Take a fluid element thin enough in the direction of the wave's propagation that it has uniform density and internal pressure (though its density and pressure change as it becomes part of a compression or rarefaction). Let ρ be the fluid's density when not disturbed by a wave passing through, and let the element's original width be dz. Each fluid element has an initial location z along the length of the pipe, and an element's displacement $s(z,t)$ at time t from its original location z varies periodically in z and in t. That is the wave. The pipe has cross-sectional area A. The fluid element's front and back walls will be shifted from their equilibrium positions as a wave passes through it. The change in the element's length will be the change in its front wall's position minus the change in the rear wall's position: $s(z + dz) - s(z)$. The change dV in its volume will therefore be $A\,[s(z + dz) - s(z)]$, and since $B = -dP/(dV/V)$, we have

$$dP = -B\,dV/V = -BA[s(z + dz) - s(z)]/A\,dz = -B\,\partial s/\partial z,$$

and so

$$\partial P/\partial z = -B\,\partial^2 s/\partial z^2.$$

The force pressing inward on the element's front wall is $A\ P(z + dz)$ in the $-z$ direction, and the force pressing inward on the element's rear wall is $A\ P(z)$ in the $+z$ direction, so the net force F on the element is $A\ [P(z) - P(z + dz)]$, and so $\mathrm{d}F/\mathrm{d}z = -A\ \partial P/\partial z$. Hence,

$$dF/dz = AB\,\partial^2 s/\partial z^2.$$

By Newton's second law, the force dF on an element is given by its acceleration $\partial^2 s/\partial t^2$ times its mass $dm = \rho A\,dz$, and so

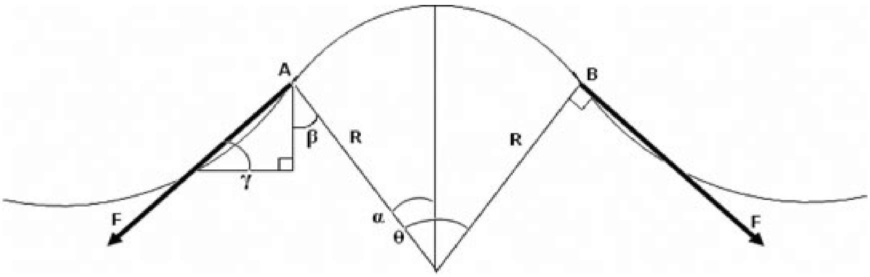

Figure 4

$$dF/dz = \rho A\,\partial^2 s/\partial t^2.$$

Equating our two expressions for dF/dz, we find

$$\rho A\,\partial^2 s/\partial t^2 = AB\,\partial^2 s/\partial z^2,$$

and so

$$\partial^2 s/\partial t^2 = (B/\rho)\,\partial^2 s/\partial z^2.$$

This is a form of the wave equation, relating the way $s(z,t)$ changes over time at a given place to the way $s(z,t)$ changes over place at a given time. It is solved by a wave propagating in the $+z$ direction at speed $v = \surd(B/\rho)$. So that is why a longitudinal wave's speed equals $\surd(B/\rho)$.

To explain the transverse wave's speed, take a small segment AB of the plucked string that makes an angle θ (figure 4). Approximate AB as the arc of a circle of radius R, so that AB's length is $R\theta$ and mass is $\mu R\theta$. Since angles α, β, and γ are congruent, and α is half of θ, the vertical component of the tension F at each end of the arc is $F\sin(\theta/2)$, so the two ends's contributions sum to 2 $F\sin(\theta/2)$, which can be approximated as $2F(\theta/2) = F\theta$ since θ is small. From the wave's viewpoint, the string is moving at speed v, and since the centripetal force for segment AB moving in a circle of radius R at speed v is $mv^2/R = \mu R\theta\ v^2/R = \mu\theta v^2$, we have

$$F\theta = \mu\theta v^2,$$

and so

$$v = \surd(F/\mu).$$

Thus we have explained why a transverse wave's v equals $\surd(F/\mu)$.

Each wave's v is proportional to its $\sqrt{}$(medium's elasticity/density). If this similarity in the two v equations were explained by the conjunction of these two derivations, then that similarity would amount to a coincidence. However, it is not, since a dimensional explanation reveals that the similarity arises from a common feature of the two waves. In so doing, the dimensional explanation unifies v's proportionality to $\sqrt{(B/\rho)}$ for the longitudinal wave with v's proportionality to $\sqrt{(F/\mu)}$ for the transverse wave. Take the explanans in the dimensional explanation to be that (to a sufficiently good approximation) for each wave, density, elasticity, and wavelength suffice (and may even be more than enough) to form a quantity standing in a dimensionally homogeneous relation to v. In other words, B, ρ, and the wavelength λ completely physically characterize the longitudinal-wave system, and F, μ, and λ completely physically characterize the transverse-wave system. Hence, v is proportional to $\rho^\alpha\ B^\beta\ \lambda^\gamma$ or to $\mu^\alpha\ F^\beta\ \lambda^\gamma$. Then the tables are as follows:

	v	ρ	B	λ
L	1	−3	−1	1
M	0	1	1	0
T	−1	0	−2	0

	v	μ	F	λ
L	1	−1	1	1
M	0	1	1	0
T	−1	0	−2	0

The M and T lines are identical in the two tables, and they give us two equations in two variables ($0 = \alpha + \beta$, $-1 = -2\beta$), from which it follows that v is proportional to $\sqrt{(B/\rho)}$ and $\sqrt{(F/\mu)}$, respectively. Although the L line in the table is different in the two cases, that difference makes no difference to α and β.[13]

Thus, unlike the derivations from more fundamental laws, the dimensional explanation traces the similarity (that each wave's v is proportional to its $\sqrt{}$(medium's elasticity/density)) to a feature common to the two systems. The explanandum is revealed to be independent of the physical differences between longitudinal and transverse waves.

Of course, in "unifying" the two cases, the dimensional explanation does not derive the two v equations from the very same facts. The dimensional explanation does not reveal a common explainer, since the expression for v in a given case is explained by the dimensional architecture of that particular case. The fact that there is a dimensionally homogeneous relation between v, B, ρ, and λ regarding longitudinal waves is plainly distinct from the fact that there is a dimensionally homogeneous relation between v, F, μ, and λ regarding transverse waves. The unification forged by dimensional explanations is not like the unity created by a common cause.

Nor is it like the unity created by Newton's theory of gravity in revealing that the moon is just a falling body—or that the tides, lunar and planetary motions, and falling bodies are all governed by the same equations. Longitudinal waves are not at bottom the same as transverse waves. Moreover, the unity forged by dimensional explanations is not like the unity created by Darwin's theory of natural selection (according to Kitcher 1993) in that it does not primarily involve showing various facts to be derivable through the same argument schemata or showing various questions to be answerable through the same problem-solving patterns. Although all dimensional explanations employ dimensional arguments, not all of the facts explained dimensionally are thereby unified with one another—unlike all of the facts about anatomy, physiology, biochemistry, biogeography, embryology, and so forth that are unified with one another in virtue of fitting into Darwinian selectionist histories. (Moreover, Darwin's theory unified various particular anatomical and biogeographical facts rather than various laws.)

Rather, the dimensional explanation unifies the derivative laws concerning v for longitudinal and transverse waves by identifying features common to both systems that account for the similarities between the two v laws. The dimensional explanation explains why the two v expressions are so similar: because of the dimensional architecture that the two systems share. To conjoin derivations of the two v laws from more fundamental laws may be to explain the two v laws, but the conjunction fails to explain why they are so similar since it inaccurately depicts their similarity as a coincidence.[14] Instead, their similarity arises from common features of the fundamental laws as they apply to the two systems—features captured by the explanans in the dimensional explanation.

A dimensional explanation, then, may identify the features shared by some physically disparate cases that are responsible for their similarity in various other respects. It may likewise identify the differences between physically disparate cases that are responsible for various other differences between them. For example, consider a simple pendulum (a small, heavy body of mass m suspended near Earth's surface by a cord of invariable length l and negligible mass and cross-section) in vacuo. The period T of its swing, when it has been released from a small angle, is (to a sufficiently good approximation) proportional to $\sqrt{(l/g)}$. Compare the simple pendulum to the system depicted in figure 5: a spring of negligible mass hangs vertically near Earth's surface (where every freely falling body experiences the same downward acceleration g from gravity), and we suddenly attach a body of mass m to the spring. The spring stretches, and then the body moves up and down for a little while, where this oscillation's period T is proportional to $\sqrt{(m/k)}$. We might contrast these two and ask: Why is the pendulum's T a function of g whereas T in figure 5's system is independent of g?

One way to try to answer this question is to derive the two T's from more fundamental laws. But this putative explanation fails to trace this particular difference between the T's to some other particular difference between the two cases. Regarding figure 5's system, we find $ma + kx - mg = 0$, and so the

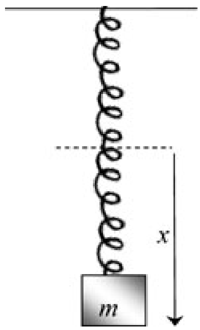

Figure 5

governing differential equation there is $a + (k/m)\,x - g = 0$, which is simple harmonic motion with period $2\pi\,\sqrt{(m/k)}$. Regarding the simple pendulum, we find $ma + [mg/l]\,x = 0$, and so the governing differential equation there is $a + (g/l)\,x = 0$, which is simple harmonic motion with period $2\pi\,\sqrt{(l/g)}$. According to this putative explanation, the pendulum's period depends on g, whereas the period of figure 5's system does not, simply because different forces are at work in the two cases. No more specific points of contrast are held responsible.

However, in dimensional terms, the two cases are more readily contrasted. On the left is our table for the pendulum[15] and on its right is our table for figure 5's system:

	T	*g*	*l*	*m*
L	0	1	1	0
M	0	0	0	1
T	1	–2	0	0

	T	*g*	*k*	*m*
L	0	1	0	0
M	0	0	1	1
T	1	–2	–2	0

The differences are confined to the third column. However, that is enough to make a considerable difference. In figure 5's system, T cannot depend on g because g is the only characteristic of the situation with an L dimension, so nothing is available to compensate for g's L dimension to leave us with pure T (for the period). In contrast, the pendulum has its characteristic l to compensate for g's contribution to the L dimension. Here we have an answer to our why-question that separate derivations from more fundamental laws cannot supply.

Likewise, why does T depend on m in figure 5's system, but not for a pendulum? Because m is the only parameter characterizing the pendulum that has an M dimension, whereas in figure 5's system, k has an M dimension that can compensate for m's contribution. Separate derivations of T in the two cases from more fundamental laws fail to point out that as long as m remains the only characteristic with an M dimension, m cannot take any responsibility for T. As we saw in connection with the $T \propto r^{3/2}$ law, derivations from the actual more fundamental laws fail to identify the specific features of these laws that are responsible for the derivative law.

5. Dimensional Explanations Can Target New Explananda

Consider the respects of similarity and difference that (we have just seen) dimensional explanations do a nice job of explaining—for example, that v is proportional to $\sqrt{}$(medium's elasticity/density) for both longitudinal and transverse waves, or that T depends on m in figure 5's system but not for a pendulum. It is easy to recognize these respects of similarity and difference. To do so, we do not need to think about the systems in dimensional terms. However, along with giving us new answers to why-questions that present themselves independently of dimensional considerations, dimensional reasoning also suggests new why-questions to ask. That is because dimensional considerations pick out new respects of similarity and difference for us to ask about.[16]

For example, consider a body at rest that begins to fall freely. In time t, it covers a distance $s = \frac{1}{2}\, gt^2$. That is, $t = \sqrt{(2s/g)}$, so t is proportional to $\sqrt{(s/g)}$. A freely falling body is physically not much like a pendulum. For instance, the falling body's motion is one-way, whereas the pendulum moves periodically; t in the above equation is not the period of any repeated motion. Yet dimensionally, t is like the pendulum's period T. Indeed, there is a dimensional similarity between the two systems: just as for a freely falling body, t is proportional to $\sqrt{(s/g)}$, so for a pendulum, T is proportional to $\sqrt{(l/g)}$.

Dimensional reasoning reveals this similarity to be no coincidence. Suppose that t for falling bodies stands in a dimensionally homogeneous relation to g, s, and m, and suppose also that T for pendulums stands in a dimensionally homogeneous relation to g, l, and m. Then the two systems have the same dimensional architecture, so dimensionally similar relations must hold among their parameters. In tabular form: the pendulum is on the left and the freely falling body is on the right:

	T	g	l	m
L	0	1	1	0
M	0	0	0	1
T	1	−2	0	0

	t	g	s	m
L	0	1	1	0
M	0	0	0	1
T	1	−2	0	0

Because the two systems have the same dimensional architecture, the derivative laws for the falling body's t and the pendulum's T must be dimensionally analogous.

A water wave having wavelength λ (traversing deep water, taking water to be incompressible and nonviscous) is physically neither like a falling body nor like a pendulum. Yet its period T is proportional to $\sqrt{(\lambda/g)}$.[17] We could ask why the derivative law for the water wave's period is so similar to the laws for the pendulum's T and the falling body's t, despite the physical differences among these phenomena. Once again, this similarity among the three derivative laws can be appreciated only in dimensional terms.

The dimensional explanation of the water wave's period starts by taking the gravitational acceleration (reflecting the restoring force on the water displaced in the wave), the water's density ρ, and the wavelength as sufficient to form a quantity standing in a dimensionally homogeneous relation to the wave's period. In tabular form:

	T	g	λ	ρ
L	0	1	1	−3
M	0	0	0	1
T	1	−2	0	0

Although this table is not perfectly dimensionally identical to the previous two, its single difference from them (in the L dimension of the only characteristic with an M dimension) makes no difference: Because that characteristic is the only one with an M dimension, it cannot figure in the dimensionally homogeneous relation. Since the three tables are otherwise identical, the resulting relations must be dimensionally analogous.

In view of the dimensional similarity among the three systems, it is inevitable that the expressions for the freely falling body's t, the pendulum's T, and the water wave's T are analogous. But that analogy can be appreciated only in dimensional terms; s, l, and λ are not otherwise analogous. That stands in contrast to the longitudinal and transverse wave example (in section 4), where the analogy between ρ and μ, and between B and F, could be appreciated without invoking dimensional considerations. (Indeed, the analogs do not have the same dimensions.)

6. Dimensional Homogeneity

In a dimensional explanation, the explanans is that there exists a dimensionally homogeneous relation between a given quantity and a subset of certain other quantities—a relation involving no other quantities besides (some of) those. A relation is "dimensionally homogeneous" if and only if the relation holds in any system of units for the fundamental dimensions of the quantities so related. Not all relations are dimensionally homogeneous. For example, that (right now) my son's weight equals my age is not dimensionally homogeneous; this relation holds only if my son's weight is measured in pounds and my age is measured in years.

This example illustrates the falsehood of remarks like this, from a standard physics textbook: "Any equation must be *dimensionally consistent*; that is, the dimensions on both sides must be the same." (Resnick, Halliday, and Krane 1992: 9). On the contrary, the numerical value of one quantity, measured in one unit, can be equal to the numerical value of another quantity, measured in another unit. It might be suggested that "equation" in the above remark refers only to laws of nature, not mere accidents, and that (as Luce (1971: 157) declares) any law of nature must be dimensionally consistent.[18] But that is not so; for example, if a body starting at rest at $t = 0$ moves by time t a distance s under uniform acceleration a, then (where v is its speed at t) $v^2 - 2as = v - at$. The two sides have different dimensions but are numerically equal, as a matter of natural law, no matter what units are used throughout for distance and time.[19] Both sides equal zero—as does $F - ma$, in yet other dimensions. Dimensional consistency is not necessary for dimensional homogeneity.[20]

Although not every relation is dimensionally homogeneous, every relation can be transformed into a dimensionally homogeneous relation by adding further relata. For example, that my son's weight in pounds equals my age in years entails that my son's weight in any unit, times the reciprocal of the number of those units in 1 pound, equals my age in any unit, times the reciprocal of the number of those units in 1 year. This relation is dimensionally homogeneous, but it involves not only my son's weight and my age, but also two "dimensional constants." (They could be combined into one.) Familiar dimensional constants appearing in natural laws include the speed of light c and Newton's gravitational-force constant G. The explanans in a dimensional explanation specifies all of the quantities, including dimensional constants, that are eligible to participate in some relation. So the explanans (that there is a dimensionally homogeneous relation between quantity Q and *no other quantities besides* a subset of the following quantities…) is not a trivial truth. But it is a trivial fact that among the relations that obtain, some are dimensionally homogeneous.

The fact that any relation can be transformed into a dimensionally homogeneous relation undermines any remark along these lines: "The fact that practically every law of physics is dimensionally invariant is certainly no accident. We need an explanation of this invariance…" (Causey 1967: 30; for similar remarks, see Causey 1969: 256; Luce 1971: 151, and Krantz, Luce, Suppes, and Tversky 1971: 504–6). We need neither to appeal to some metaphysical analysis of natural lawhood nor to posit some meta-law of nature (e.g., that it is a law that all laws are dimensionally homogeneous)[21] in order to explain why all actual laws involve dimensionally homogeneous relations. Rather, we need note only that any law expressed in terms of a relation that is not dimensionally homogeneous can also be expressed in terms of a relation that is dimensionally homogeneous.[22] For instance, in middle school I was taught that it is a law of nature that the distance s in feet traversed in t seconds by a body falling freely from rest is equal to $16t^2$. This law is plainly not dimensionally homogeneous. But if we replace the pure number 16 with the dimensional constant 16 feet per second^2, then we will have a dimensionally homogeneous relation.[23]

In any event, a particular dimensional explanation does not presuppose some general principle that all laws involve dimensionally homogeneous relations. Rather, the explanans in a given dimensional explanation is much narrower: that there obtains a relation between a given quantity and a subset of several other quantities that is dimensionally homogeneous for a certain set of fundamental dimensions. This reference to certain dimensions is the next topic I shall discuss.

7. Independence from Some More Fundamental Laws May be Part of a Dimensional Explanans

In a dimensional explanation, the explanans is that certain quantities suffice to produce a dimensionally homogeneous relation—which entails that certain other quantities do *not* need to be added. Whereas a deduction of some derivative law from more fundamental laws may appeal to a law that specifies the value of a given dimensional constant, for example, a dimensional explanation of that derivative law may appeal to the fact that the explanandum would still have held whatever that dimensional constant's value.

Consider, for instance, a small sphere of radius r falling slowly through a fluid of viscosity η. It quickly reaches a steady speed, its "terminal velocity," which turns out to be proportional to gr^2/η. This fact may be derived hydrodynamically or dimensionally.

The hydrodynamic derivation begins with Newton's second law of motion: the net force on the sphere equals its ma. When terminal velocity is reached, $a = 0$. The net force on the sphere is the sum of the downward gravitational force F_g, the upward buoyant force F_b, and the upward drag force F_d that increases with the sphere's speed. By Archimedes's principle, F_b equals the weight of a fluid volume equal to the body's volume. Furthermore, F_d is given by Stokes's law when the fluid flow around the body is not turbulent (roughly, for a small body at low speed). If the fluid's density is ρ_{f} and the sphere's density is ρ_{s}, then

$$
\begin{aligned}
F_g &= mg = \rho_s(4/3)\pi r^3 g \\
F_b &= -\rho_{\mathrm{f}}(4/3)\pi r^3 g \\
F_d &= -6\pi r v \eta.
\end{aligned}
$$

So

$$(4/3)\pi gr^3(\rho_s - \rho_f) = 6\pi rv\eta.$$

Hence

$$v = (2/9)gr^2(\rho_s - \rho_f)/\eta.$$

Thus, we have derived that v is proportional to gr^2/η.

Now consider a dimensional explanation of this law. Suppose the explanans is that ρ_s, ρ_f, r, g, and η suffice to characterize the system physically. Hence (since ρ_f/ρ_s is dimensionless, and so its contribution is inaccessible from exclusively dimensional considerations), there is a dimensionally homogeneous relation $v = r^\alpha \ \rho_s{}^\beta \ \eta^\gamma g^\delta \ f(\rho_f/\rho_s)$ for some function f.[24] If this relation is dimensionally homogeneous in terms of three dimensions (such as L, T, and M), then unhappily, a table yields three simultaneous equations in four variables (α, β, γ, and δ). However, suppose (as Bridgman 1931: 66 suggests) that this relation is dimensionally homogeneous in terms of four dimensions: L, T, M, and force (F). Notice that according to this explanans, the dimensional constant of proportionality k between force and mass x acceleration ($F = kma$) is *not* among the quantities standing in the dimensionally homogeneous relation. It bears no explanatory responsibility for the fact that v is proportional to gr^2/η because whatever k's value, that value is the same for all three component forces, so the value does not affect the conditions under which they are balanced, which is the condition with which we are concerned. No force is unbalanced, so no force is producing acceleration, so the rate of exchange between force and *ma* does not matter.[25] Then we have four simultaneous equations in four variables. Our table is

	v	r	ρ_s	η	g
L	1	1	−3	−2	0
M	0	0	1	0	−1
T	−1	0	0	1	0
F	0	0	0	1	1

Our four equations are then

From L: $1 = \alpha - 3\beta - 2\gamma$

From M: $0 = \beta - \delta$

$$\text{From T:} \quad -1 = \gamma$$

$$\text{From F:} \quad 0 = \gamma + \delta.$$

Hence: $\alpha = 2$, $\beta = 1$, $\gamma = -1$, $\delta = 1$, yielding $v = (r^2 \rho_s g/\eta)\, \mathrm{f}(\rho_f/\rho_s)$.[26] Thus, we have explained why v is proportional to gr^2/η.

This equation is thus explained by the fact that there is a relation between v, ρ_s, ρ_f, r, g, and η that is dimensionally homogeneous *using these four dimensions* and so holds for any value of k. Whereas $F = ma$ launches the hydrodynamic derivation, the dimensional explanation not only fails to employ $F = ma$, but actually employs the fact that the dimensionally homogeneous relation would still have held whatever the value of k where $F = kma$. In the dimensional explanation, part of the explanans is that the explanandum is independent from a certain feature of a more fundamental law that is used to deduce it hydrodynamically. In other words (putting the point more provocatively), part of the explanans is a counterfactual (indeed, a counterlegal): that the explanandum would still have held, for any value of k.[27]

This aspect of some dimensional explanations is crucial to resolving a venerable dispute. In an early discussion of dimensional methods, Rayleigh (1915a) considered the rate at which heat passes from a hot wire to a cooler stream of air passing across it—or, idealizing and generalizing, the rate h of heat loss by a rigid body of infinite conductivity and presenting a linear dimension a to an ideal (i.e., incompressible, inviscid) fluid flowing at speed v around it, where the body's temperature is kept constant and exceeds the fluid's initial temperature (i.e., far upstream from the body) by θ. Let c be the fluid's specific heat per unit volume (i.e., the heat needed to raise the temperature of a unit volume of fluid by one degree) and let κ be the fluid's thermal conductivity (i.e., the rate of heat flow through a unit thickness of the fluid per unit area and unit temperature difference). Rayleigh used dimensional analysis to arrive at an expression for h, though I suggest that we consider him as giving a dimensional explanation of that expression. His proposed explanans is that h stands in a dimensionally homogeneous relation to some subset of a, v, θ, c, and κ, where the fundamental dimensions are L, T, Θ (temperature), and Q (heat). If h is to have the same dimensions as $a^\alpha\, v^\beta\, \theta^\gamma\, c^\delta\, \kappa^\varepsilon$, and our table is

	h	a	v	θ	c	κ
L	0	1	1	0	−3	−1
T	−1	0	−1	0	0	−1
Θ	0	0	0	1	−1	−1
Q	1	0	0	0	1	1

then (Rayleigh argued) we have four simultaneous equations with five unknowns

$$\text{From L:} \quad 0 = \alpha + \beta - 3\delta - \varepsilon$$

$$\text{From T:} \quad -1 = -\beta - \varepsilon$$

$$\text{From } \Theta: \quad 0 = \gamma - \delta - \varepsilon$$

$$\text{From Q:} \quad 1 = \delta + \varepsilon$$

yielding $\alpha = \beta + 1$, $\gamma = 1$, $\delta = \beta$, and $\varepsilon = 1 - \beta$. Therefore, h has the same dimensions as $a^{\beta+1}\ v^{\beta}\ \theta\ c^{\beta}\ \kappa^{1-\beta}$, i.e., as $(avc/\kappa)^{\beta}\ a\theta\kappa$, and so $h = a\theta\kappa\ \mathrm{f}(avc/\kappa)$ for some unknown function f. We have thereby explained why the rate of heat loss is proportional to the temperature difference, why it is the same in cases involving different values of v and c but where vc is the same, and so forth.

However, in an economical three-sentence reply, Riabouchinsky (1915) noted that temperature and heat have the same dimensions as energy, so that we have here only three fundamental dimensions, not four. (The average kinetic energy of random motion per molecule is related to temperature by Boltzmann's constant, equal to 1.38×10^{-16} ergs/degree Kelvin, and the rate of exchange between heat and energy units is given by the mechanical equivalent of heat, equal to 4.184 Joules/calorie.) We thus have one fewer equation, but the same number of unknowns, so dimensional analysis yields less information—merely that $h = a\theta\kappa\ \mathrm{F}(v/\kappa a^2, ca^3)$, for some unknown function F. (This dimensional argument fails to explain why heat loss is the same in cases where vc is the same, for instance.) Rayleigh (1915b) replied uncertainly:

> The question raised by Dr. Riabouchinsky belongs rather to the logic than to the use of the principle of similitude, with which I was mainly concerned. It would be well worthy of further discussion. The conclusion that I gave follows on the basis of the usual Fourier equations for conduction of heat, in which heat and temperature are regarded as *sui generis*. It would indeed be a paradox if the further knowledge of the nature of heat afforded by molecular theory put us in a worse position than before in dealing with a particular problem. The solution would seem to be that the Fourier equations embody something as to the nature of heat and temperature which is ignored in the alternative argument of Dr. Riabouchinsky.

Although commentators have generally been hard on Rayleigh (Bridgman (1931: 11), for example, says that Rayleigh's reply "is, I think, likely to leave us cold"), Rayleigh is essentially correct. His dimensional argument (whether used as prediction or explanation) takes as a premise that the target equation for h is dimensionally homogeneous *using* L*,* T*,* Θ*, and* Q *as fundamental* *dimensions*, just as in the case of the sphere falling through the viscous fluid, the explanans was that v stands in a relation to ρ_s, ρ_f, r, g, and η that is dimensionally homogeneous using L, M, T, and F as fundamental dimensions. This is indeed further information beyond the fact that the target equation for h is dimensionally homogeneous using L, M, and T (or equivalently L, T, and energy). This is not a case where more information puts us in a worse position than before. Rather, more information puts us in a better position than before.

In the dimensional explanation of the derivative law governing the falling sphere's terminal velocity, the explanans includes that the law would still have held for any value of the constant of proportionality between force and *ma*. Likewise, in Rayleigh's dimensional explanation of the heat-flow equation, the explanans includes that the equation would still have held had (counterlegally) Boltzmann's constant and the mechanical equivalent of heat departed from their actual values—or (counterlegally) had there been no such constants at all: had heat been a fluid (as caloric was thought to be). In the falling-sphere case, the rate of exchange between force and acceleration is explanatorily irrelevant because no force is producing an acceleration; the sphere has reached its terminal velocity. In the heat-flow case, the rate of exchange between heat and energy, or between temperature and random molecular kinetic energy, is explanatorily irrelevant because no thermal energy is being converted into mechanical energy (or any other sort of energy) or vice versa. For example, the fluid is not being called upon to do work (e.g., by pushing against a piston) and no internal molecular energy is being transformed into thermal energy. The thermal energy of the rigid body is simply contributing to the fluid's thermal energy.[28]

In what I believe he intends to be an allusion to the Rayleigh-Riabouchinsky exchange, Bridgman (1931: 49–50) writes:

> [H]ow [shall we] choose the list of physical quantities between which we are to search for a relation[?] We have seen that it does not do to merely ask ourselves "Does the result depend on this or that physical quantity?" for we have seen in one problem that although the result certainly does "depend" on the action of the atomic forces, yet we do not have to consider the atomic forces in our analysis, and they do not enter the functional relation.

Here Bridgman appears to distinguish dimensional analysis from scientific explanation (perhaps with scare quotes), whereas I regard dimensional analysis as sometimes supplying explanations of derivative laws. In a derivation of the heat-flow equation from much more fundamental laws, the molecular nature of heat is invoked. However, the heat-flow equation (up to the unspecified function $f(avc/\kappa)$) can be accounted for by a dimensional explanation. The equation considers the phenomenon of heat flow purely at a phenomenological level (e.g., in employing c and κ, which are properties of bulk matter), and (as Sedov 1959: 42 emphasizes) no thermal energy is converted into other forms of energy or vice versa (since the fluid is ideal). Therefore, it is not evident that the equation actually depends on the molecular nature of heat. Indeed, part of the explanans in the dimensional explanation is precisely that the equation relating these various quantities to heat flow would still have held, even if (say) there had been no mechanical equivalent of heat.

8. Some Dimensional Explanations May Invoke Dimensional Meta-Laws

The explanans in a dimensional explanation is the fact that in order to form a quantity standing in a dimensionally homogeneous relation to a given quantity, we need use nothing more than certain quantities (in the proper combination); we need no other quantities besides (some of) those. Like the explanans of a dimensional explanation, a symmetry principle also specifies certain quantities to be dispensable to certain laws. For instance, the principle of symmetry under arbitrary spatial displacement specifies that the location of any given event can be omitted from any law. Only the separation between events matters to the law; the law privileges no particular spatial location. Space is "homogeneous."

Such symmetry principles are frequently characterized as "meta-laws"—that is, as laws governing the first-order laws and thereby explaining why they have certain features. For example, Wigner characterizes a symmetry principle as "a superprinciple which is in a similar relation to the laws of nature as these are to the events" (1972: 10) and "as laws which the laws of nature have to obey" (1985: 700; for similar remarks, see Morrison 1995 and Feynman, 1967: 59, 83). In view of space-displacement symmetry, the invariance of every first-order law under arbitrary spatial displacement is not a mere coincidence—a feature that every first-order law independently just happens to possess. Rather, there is a common explanation of each first-order law's exhibiting this symmetry: each does so because all first-order laws have to, as a matter of meta-law.

Indeed, symmetry meta-laws, as constraints on the first-order laws, are widely taken as helping to account for various first-order laws, such as the conservation laws of momentum and energy.[29] Since a symmetry meta-law "stands on its own, independent of any detailed theory of" the particular kinds of forces there are (Weinberg 1992: 158), the first-order laws explained by symmetry meta-laws hold independently of those details. For example, energy and momentum would still have been conserved even if (here comes a counterlegal) the particular forces had been different. As Wigner remarks,

> [F]or those [conservation laws] which derive from the geometrical principles of invariance it is clear that their validity transcends that of any special theory—gravitational, electromagnetic, etc.—which are only loosely connected . . . (Wigner 1972: 13)

Thus, although energy and momentum conservation in a given system can be deduced in classical physics from Newton's laws of motion and the laws governing the fundamental forces operating in that system, this deduction does not correctly explain why energy and momentum are conserved, since the details of the various particular force laws are explanatorily irrelevant. Energy's conservation is not a coincidental byproduct of the sundry force laws.

As we have seen, dimensional explanations likewise reveal various derivative laws to be independent of various details of the more fundamental laws from which they can be deduced. This similarity suggests that perhaps some dimensional explanations proceed just like explanations from symmetry principles in that the explanans consists of meta-laws: principles transcending the first-order laws and imposing restrictions on the kinds of first-order laws there could have been.

To find a possible example, we must consider a derivative law that is plausibly explained entirely by meta-laws. Consider, for instance, the coordinate transformation laws, which have standardly been interpreted as purely kinematical—that is, as entirely independent of the particular kinds of forces there happen to be.[30] Let us suppose that classical physics holds, and so the coordinate transformation laws employ the Galilean transformations. Consider two inertial reference frames S and S′ (figure 6) where corresponding axes are parallel and where the origin O′ in S′ moves relative to O (the origin in S) with constant speed v in the x direction. Suppose that the moment when O coincides with O′ is the reference event marking time zero for both reference frames. That is, their coinciding is an event with (x,y,z,t) coordinates (0,0,0,0) and with (x′,y′,z′,t′) coordinates (0,0,0,0). Then according to classical physics[31], the laws transforming any event's unprimed into primed coordinates are $t' = t$, $z' = z$, $y' = y$, and $x' = x - vt$.

I shall set out a dimensional explanation of why it is that in the x' law, v and t appear only to the first power and only together—that is, a dimensional explanation of why there is no term in which v figures without t, or vice versa,

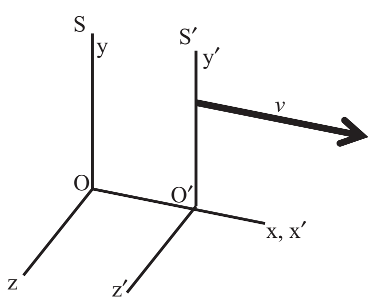

Figure 6

or (for example) involving v^2 or $\sqrt{t}$. Our explanation involves both symmetry meta-laws and dimensional meta-laws, and so shows that the explanandum transcends any details of the first-order laws that are not required by meta-law.[32]

Let us start our explanation. If the transformation yields any event's x' as a function f of various quantities, then what quantities could they be? Take the dimensional meta-law to be that v and the event's x, y, z, and t suffice to form a quantity that stands in a dimensionally homogeneous relation f to the event's x'.[33] That transformations of the same form hold between all pairs of inertial frames follows from the "principle of relativity": that the natural laws take the same form in all inertial frames (a meta-law). Furthermore, the "homogeneity of space" imposes a further demand on the first-order laws: that they take the same form in any two reference frames differing only in the location of their origin. Hence, the transformation from S to S′ must also work for transforming from S″ into S′, where S″ differs from S only in its origin being displaced from O by some distance along the y axis. Hence, if y figures in the transformation from x to x', then y'' must likewise figure in the transformation from x'' to x'. But O′ has the same speed v relative to O as it does relative to O″ and any event's (x,y,z,t) will be the same as its (x″,y″,z″,t″) except for its y coordinate differing from its y'' coordinate. Therefore, if y figures in the transformation from x to x', and if v, x, y, z, and t are the only quantities used by that transformation to yield x', then the transformation from S will yield a different x' value for a given event than the transformation from S″ does. Therefore, y cannot figure in the transformation from x to x' (and, by similar reasoning, neither can z).

Without elaboration, Einstein (1905/1952: 44) appeals to the homogeneity of space to show in addition that the transformation must be linear in x (that is, no term in the transformation can raise x to higher than the first power). Here is one way that Einstein's reasoning could be unpacked. Consider a rod placed at time t along the x-axis so that in S, its left endpoint is at x_1 and its right endpoint is at x_2. In S, its length is $x_2 - x_1$; in S′, its length is $f(x_2,v,t) - f(x_1,v,t)$. Now instead suppose the same rod had been placed with its left endpoint shifted to $x_1 + \Delta$ in S. By the homogeneity of space, the rod would have had the same length in S, so its right endpoint would have been at $x_2 + \Delta$ in S. In S′, then, its endpoints would have been at $f(x_1 + \Delta,v,t)$ and $f(x_2 + \Delta,v,t)$. By the homogeneity of space, its length in S′ would have been unaffected by the shift:

$$f(x_2 + \Delta, v, t) - f(x_1 + \Delta, v, t) = f(x_2, v, t) - f(x_1, v, t).$$

Hence

$$f(x_2 + \Delta, v, t) - f(x_2, v, t) = f(x_1 + \Delta, v, t) - f(x_1, v, t).$$

Dividing both sides by Δ, and then taking their limits as $\Delta \to 0$, we find

$$\partial f/\partial x(\mathrm{x}_2) = \partial f/\partial x(\mathrm{x}_1).$$

That is, f's partial derivative with respect to x is the same whether evaluated at x_1 as at x_2, for arbitrary x_1 and x_2. So $\partial f/\partial x$ is a constant. Therefore, f must be linear in x.

Since x' is to have the same dimensions as $x^\alpha\ t^\beta\ \mathrm{v}^\gamma$, our table is

	x'	x	t	v
L	1	1	0	1
T	0	0	1	−1

From L: $1 = \alpha + \gamma$

From T: $0 = \beta - \gamma$

Because the transformation must be linear in x, α can only be 0 or 1. If a term in $f(x,t,v)$ has $\alpha = 1$, then $\beta = \gamma = 0$. If a term has $\alpha = 0$, then $\beta = \gamma = 1$. Therefore, any term in $f(x,t,v)$ must be either proportional to x (without v, t) or proportional to vt (without x): the transformation law must be $x' = \mathrm{A}x + \mathrm{B}vt$ (for some dimensionless constants A, B).

Plausibly, this dimensional explanation uses a dimensional meta-law alongside a symmetry meta-law to explain a feature of the transformation laws. In view of being explained by these meta-laws, this feature transcends the particular force laws and other first-order laws there happen to be.

9. Conclusion

Dimensional explanations of derivative laws illustrate the fact that a derivative law need not be explained by its deduction from more fundamental laws. Similar phenomena arise in connection with explanations in mathematics (see Lange forthcoming). Here is a very simple example. If we have three objects and each object is red or blue, then two of the objects are the same color. Why is that? We can deduce this fact by listing all 8 possibilities (RRR, RRB, RBR, RBB, BRR, BRB, BBR, BBB) and noting that in each case, at least two of the objects are the same color. But this derivation fails to explain why the result obtains; for each possibility, there is the same reason why it has two objects of the same color. Select one object. Either it is the same color as one of the others, or it is not. If it is not, then (since there are only two possible colors) the other two objects must be the same color.

This explanation shows that the explanandum (unlike the 8 possible combinations) is not sensitive to various details of the case (such as the particular pair of colors involved). Indeed, the same sort of mathematical explanation can be given of the fact that if we have four objects, and each of them is red, white, or blue, then two of the objects are the same color. (The analogous explanation: Select an object. Either it is the same color as one of the others, or it is not. If not, then select another object. We know that it is not the same color as the object first selected, and either it is the same color as one of the remaining unselected objects, or it is not. If not, then since there are only three colors, the two remaining objects must be the same color.)

Plainly, it is no coincidence that there is color duplication in both of these cases: both involve more objects than colors.[34] But the two separate, brute-force deductions of color duplication from the two lists of every possibility in the two cases fails to remove the coincidence; that pair of deductions does not unify the two cases by tracing the color duplication in each case back to a feature that is also possessed by the other case. The two lists of every possibility in two such cases fail to pick out the particular features of the two cases that are responsible for a given similarity or difference between them. For instance, the two lists fail to explain why two of the objects must be the same color when there are three objects and two possible colors, but not when there are three objects and three possible colors.

Even after a theorem in mathematics has been well-established, mathematicians labor to discover new proofs of it partly because of the new explanatory connections that those proofs may reveal. A similar phenomenon occurs in science. Dimensional explanations exemplify some of the kinds of explanatory contributions that can be made by different styles of scientific reasoning.[35]

Notes

[1] Among the philosophical works on dimensional analysis are Bridgman (1931), Campbell (1957), Causey (1967, 1969), Ellis (1966), Krantz, Luce, Suppes, and Tversky (1971), Luce (1971), and Laymon (1991). None characterizes dimensional analysis as capable of funding scientific explanations. However, a reviewer kindly called my attention to Batterman's work (e.g., Batterman 2002a, 2002b), where the explanatory contributions of dimensional analysis *are* explored. Batterman understands dimensional analysis as the simplest species of "asymptotic explanation", and Batterman sees an asymptotic explanation as supplying a kind of understanding that cannot be supplied, even in principle, by a derivation of the same explanandum from more fundamental laws. There are thus close similarities between Batterman's themes and mine.

Batterman's "asymptotic explanations" target relationships that emerge in the long run for a wide range of specific initial conditions (as the influence of those initial details dies out) and at the macroscopic level (where various microstructural details make negligible difference). Like Batterman, I emphasize how dimensional explanations unify because they can ignore certain physical details. However, although (as we shall see) dimensional arguments work by presupposing that (at least to a sufficiently good approximation) a given phenomenon is independent of various parameters, I contend neither that an equation explained dimensionally must be the result of some more fundamental theory taken to some limit nor that a dimensional argument is explanatory by virtue of capturing the asymptote of some fundamental theory.

In addition, whereas Batterman emphasizes that the same asymptotic explanation applies to (for instance) all simple pendulums whatever material their bobs are made of, I argue that dimensional explanations allow connections to be drawn among much more disparate cases—for instance, among a pendulum, a body in free fall, and a water wave (see section 5). These cases have the same (or relevantly similar) dimensional architecture despite differing profoundly in their physical features *even when considered asymptotically*—in the limit where the influences of initial transients and microstructural details become negligibly small.

[2] By a "derivative law", I mean a logical consequence of natural laws alone (together with mathematical truths and other broadly logical necessities) that is not a fundamental law, but rather is explained by other laws. For example (according to classical physics), the centripetal force law $F = m\omega^2 r$ follows from and is explained by Newton's second law of motion (together with geometric facts). Some philosophers use the term "derivative law" to encompass some generalizations that do not follow from natural laws alone. For example, that all bodies falling to Earth from a small height in conditions where all non-gravitational influences can be neglected (for example, with negligible air resistance) accelerate at approximately 9.8 $\mathrm{m/s^2}$ is sometimes termed a "derivative law" ("Galileo's law of falling bodies") even though it follows from more fundamental laws (such as Newton's law of gravity and second law of motion) only when they have been supplemented by various non-laws, such as the accidental fact that Earth's mass is 5.98×10^{24} kg. I shall be concerned only with "derivative laws" that are naturally necessary rather than accidental. A given derivative law's scope may be rather narrow; the law may be a consequence of more fundamental laws applied to a certain specific range of conditions.

[3] In my forthcoming, I argue that when two derivative laws of nature are similar, despite concerning physically distinct processes, it may be that any correct scientific explanation of their similarity proceeds by revealing their similarity to be no *mathematical* coincidence. Their similarity is explained mathematically—by mathematical similarities between the more fundamental equations responsible for the two derivative laws.

[4] That this derivation is generally regarded as explanatory is evident from common remarks along these lines: "In Newton's work the inverse square law appears as a means of accounting for the observations of the solar system, particularly Kepler's interpretation of Brahe's observations in terms of a relationship between periods and radii of orbits." (Weinberg 1987: 6). But we will have to consider whether the fact that this derivation explains why $T^2 = 4\pi^2 r^3/GM$ suffices to ensure that it explains why $T \propto r^{3/2}$.

[5] Unfortunately, there are several common definitions of the term "dimensionally homogeneous". One is the definition that I just gave: that a relation is "dimensionally homogeneous" if and only if it holds in any system of units for the fundamental dimensions of the quantities so related. This definition may be found in articles and textbooks concerning a wide range of subjects—for example, coastal engineering (Hughes 1993: 27), engineering experimentation (Schenck 1979: 88), fluid mechanics (Shames 2002: 8), and pharmacokinetics (Rescigno 2003: 23), as well as dimensional analysis (Langhaar 1951: 13 and 18; Laymon 1991: 148). (I will refine this definition in section 7 so as to recognize that a given relation is dimensionally homogeneous *for a certain set of fundamental dimensions*, such as {mass, length, and time}.) A second common definition is that a relation is "dimensionally homogeneous" if and only if every term in the equation has the same dimensions. Such definitions appear in Birkhoff 1950: 80, Bridgman 1931: 41, and Furbish 1997: 116, for example. (However, Bridgman (1931: 37) also says that the assumption that a given relation holds for any system of units for the various fundamental dimensions of the quantities so related "is absolutely essential to the treatment, and in fact dimensional analysis applies only to this type of equation.") In section 6, I contrast this second notion (which I term "dimensional consistency") with the first (which I call "dimensional homogeneity"). As a third alternative, "dimensional homogeneity" is sometimes defined directly in terms of the equation's form, as described two paragraphs below in the main text.

[6] This "motivation" fails to motivate the requirement that $c > 0$, which presumably arises from the thought that two units cannot measure the *same* quantity if one increases while the other decreases (though they might then measure logically related quantities).

[7] This criterion for units to specify the same quantity occasionally appears to depart from ordinary usage. For example, degrees Celsius and degrees Kelvin are ordinarily regarded as measuring the same quantity: temperature. But since their zeros do not coincide, a doubling of degrees Celsius—e.g., from 2°C (275°K) to 4°C (277°K)—does not coincide with a doubling of degrees Kelvin. However, if °K measures how far a temperature departs from absolute zero, then °C does not count as a unit for the same quantity as °K. (In laws such as the ideal-gas law (PV = nRT), T must be expressed in an absolute scale.) Nevertheless, on this criterion, °C and °K are both units of temperature *difference*, since then the arbitrary zeros drop out.

[8] Had T depended on the planet's mass, then Kepler's third law (that $T \propto r^{3/2}$ with the same proportionality constant for every planet orbiting the sun) would not have held.

[9] There might seem to be one difference: the salt's being hexed explains nothing whereas the elements of the fundamental laws that do not contribute to explaining why $T \propto r^{3/2}$ holds help to explain other facts (e.g., T's independence from m). However, the salt's being hexed might explain how superstitious people treat the salt.

[10] Here, I think, my view disagrees with Campbell's. Although not discussing dimensional arguments as explanations, he says that "for the application of the argument from dimensions everything involved in the dynamical reasoning is required except the numerical values of no-dimensional magnitudes" (1957: 403, cf. 422).

[11] That derivation, unlike the dimensional explanation, explains why the proportionality constant between T and $\sqrt{(m/k)}$ equals 2π.

[12] Technically, this is the *adiabatic* bulk modulus (rather than the *isothermal* bulk modulus) because although the compressions and rarefactions are associated with temperature changes, they occur so rapidly that little heat can flow.

[13] The L line in each case dictates that v is independent of λ—though for different reasons in the two cases. However, we could not have reached this result had we also included the wave's amplitude in characterizing it. (But the M and T lines would have been unaffected.)

[14] That two facts are naturally necessary does not suffice to make a given consequence of them no coincidence. For instance, nineteenth-century chemists believed it naturally necessary that all noncyclic alkane hydrocarbons differ in molecular weight by multiples of 14 units, and they also believed it naturally necessary that the atomic weight of nitrogen is 14 units. But they termed it "coincidental" (albeit naturally necessary) that all noncyclic alkanes differ in molecular weight by multiples of the atomic weight of nitrogen. Noncyclic alkanes contain no nitrogen. See, for instance, van Spronsen 1969: 73–4 and Lange 2000: 203–7 as well as my (forthcoming) account of mathematical coincidences.

[15] I have omitted a column for the angle from which the pendulum is released, since angle is usually taken to be a dimensionless quantity, and so dimensional analysis cannot impose any constraint on how it figures in the equation for the period. That is, dimensional analysis reveals only that T is proportional to $\sqrt{(l/g)}$ times some unknown function of the initial angle.

[16] Similarly, Hacking (1990) emphasizes that statistical reasoning identifies new targets of explanation (e.g., normal distributions).

[17] The constant of proportionality turns out to be $\sqrt{(2\pi)}$.

[18] Likewise Barenblatt (1987: 5): "The dimensions of both sides of any equation having physical sense must be identical. Otherwise, the equation would no longer hold under a change of fundamental units of measurement." Douglas (1969: 3): "An equation about a real physical situation will be true only if all the terms are of the same kind and therefore have the same dimensions."

[19] Bridgman (1931: 42) and Birkhoff (1950: 83) give similar examples. Bridgman emphasizes that "=" in the equation should be understood as numerical equality; obviously, if "=" required the same units on both sides, then trivially an equation would have to be dimensionally consistent.

Some might say that the example, though it follows logically from natural laws alone, is not itself a law, and so fails to show that laws can be dimensionally inconsistent. In Lange 2000, I discuss the distinction between laws and naturally necessary non-laws.

[20] Nor is dimensional consistency (plus truth) sufficient for dimensional homogeneity. In cgs electrostatic units, charge (like force) is not an independent dimension. It has dimension $L^{3/2}\ M^{1/2}T^{-1}$. (One "electrostatic unit" is defined as the charge where the electrostatic force between two point bodies so charged, 1 cm apart, equals 1 dyne.) Accordingly, Coulomb's law in these units ($F = q_1\ q_2\ /\ r^2$) has no dimensional constant of proportionality. A change to other units would require the introduction of a dimensional proportionality constant. So this expression for Coulomb's law is dimensionally consistent (in cgs electrostatic units) but not dimensionally homogeneous.

[21] Shames (2002: 8), for example, invokes a "law of dimensional homogeneity" (that "an analytically derived equation representing a physical phenomenon must be valid for all systems of units") as if it were one among the many contingent laws and meta-laws of nature.

[22] In this, I think I agree with Campbell (1957: 366–9) and Ellis (1966: 117). That all laws can be expressed in terms of dimensionally homogeneous relations seems akin to the view (advanced by Reichenbach, Hempel, Carnap, and others) that all fundamental laws can be expressed without proper names or "local predicates" (i.e., predicates defined in terms of particular times, places, objects, events, etc.). Indeed, a similar intuition lies behind both ideas: the laws are too "general" to privilege any particular thing (whether an object or a unit). For more on natural laws and nonlocal predicates, see Lange 2000.

[23] Since it is trivial that every law can be expressed in terms of a dimensionally homogeneous relation, I do not understand why those who believe in some non-trivial "principle of dimensional homogeneity" qualify the principle; they say that we need an explanation "of why *(most)* numerical laws of physics are dimensionally invariant" (Krantz et al. 1971: 504, my italics) or "of the *prevalence* of dimensionally invariant laws (Causey 1969: 256, my italics) or of the fact "that *practically* every law of physics is dimensionally invariant" (Causey 1967: 30, my italics) or of the fact that "most, if not all, physical laws can be stated in terms of dimensionally invariant equations" (Luce 1971: 157).

[24] Having included ρ_s and an unspecified $f(\rho_f/\rho_s)$ in the relation, there is no need also to include ρ_f.

[25] For another example of a law that has sometimes been regarded as having a statical rather than a dynamical explanation (and so as independent of Newton's second law), see my forthcoming2 concerning the law of the parallelogram of forces.

[26] This yields the hydrodynamic answer when $f(\rho_f/\rho_s) = (2/9)\ (1 - (\rho_f/\rho_s))$.

[27] In Lange 2005, 2007 and forthcoming2, I have more extensively discussed the role of counterfactuals (and other subjunctive facts) in explaining actuals. For particular attention to counterlegals, see Lange 2009.

[28] In contrast, take a case where thermal energy *is* converted to or from another form of energy. In giving a dimensional explanation for the gain in heat h of a body having mass m upon falling to the ground (without bouncing) from rest at height s, we must presume there to be a dimensionally homogeneous relation between h and some subset of g, m, and s where heat is *not* given its own dimension. (No such dimensionally homogeneous relation is possible if heat is given its own dimension.) Dimensional reasoning then yields $h \propto mgs$.

[29] Symmetries are standardly taken as explaining conservation laws—see, for instance, Landau and Lifshitz 1976: 13; Wigner 1954: 199; Gross 1996: 14257; and Feinberg and Goldhaber 1963: 45. Not all philosophers agree that these arguments are explanatory (see, for instance, Brown and Holland 2004: 1137–1138). I do not have the space here to examine the many interesting questions about whether these arguments are genuinely explanatory and, if so, why. Lange 2007 and 2009 address these issues and specify the sense in which symmetry principles and the conservation laws they explain transcend the various particular force laws.

[30] Brown (2005) is the most recent exponent of the heterodox tradition of regarding the coordinate transformations as dynamical rather than as kinematical; Brown (2005) contains

many further references to both traditions and an account of how the Galilean transformations could be explained dynamically in classical physics—just as I shall now set out one way they could be explained kinematically.

[31] Of course, the relativistic transformation laws are the Lorentz transformations rather than the Galilean transformations. But for the purpose of showing how a dimensional explanation might proceed from meta-laws, it suffices to give an argument that could be explanatory according to classical physics.

[32] Some readers might regard the Galilean transformation laws as too obvious to require explanation. However, they are not obvious; in fact, they are false (see previous note). So it is especially revealing to see where classical physics might say they come from, since one component of that explanans must be false. This highlights the fact that neither symmetry meta-laws nor dimensional meta-laws are knowable *a priori*. Neither consists of logical or conceptual necessities.

[33] This turns out to be the premise that is violated in special relativity (see previous note); a dimensionally homogeneous relation actually requires a further dimensional constant with the dimensions of speed.

[34] In Lange forthcoming, I discuss the notion of a "mathematical coincidence" and its relation to mathematical and scientific explanation.

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